Nonequivalent Lagrangian Mechanics

A Thesis
Presented to
The Division of Mathematics and Natural Sciences
Reed College

In Partial Fulfillment
of the Requirements for the Degree
Bachelor of Arts

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May 2014

Approved for the Division
(Physics)

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Acknowledgements

In order to best acknowledge those who deserve thanks I break my acknowledgements into three divisions, academic, personal, and a dedication.

Academic:

I thank the teachers who have in the past four years changed my life and my perspective on the world. Tom Wieting, for his endlessly interesting insights into the exquisitely beautiful world of math and mathematical physics, and for the poetry he imbues in those topics, as well as for coming in as a hidden secondary advisor for my thesis when it became too much for us. Joel Franklin, for being an inspiration, showing me the door into the true grandeur of applied math, and putting up with my endless questions.

And of course, my advisor Nelia Mann. It has meant an incredible amount to me to be able spend this year working with her. She has been an inspiring teacher and corroborator. This thesis could not have been written with out her.

Personal:

I thank my family, for supporting me both financially and personally in my quest to do math and physics. I joke that it was fate that brought me to where I am at Reed, but the reality is I was born and raised into it, and for that I am forever grateful.

I thank my friends, my time with you has been overwhelmingly important to me. I thank the Lutz Wednesday crew. I thank Black House and its satellites. I thank all of my fellow physics majors, but principally Greg Kohler and Stu Pickell. I thank the Mojave desert.

I thank Allie, who has put up with all the suffering and madness that was involved with writing this thesis. I can't imagine someone better to have spent the year with.

Dedication:

Finally, this thesis is dedicated to both my maternal grandfather, Charles Hendricks, and paternal great grandmother, Ona Hardacker.

For the former, because the intensity of his brilliance is so great that he has driven both my family and myself into a better world then any of us could have possibly dreamed of.

For the latter, because when she was given the choice between a car and college education at the end of high school, picked the education, and that has defined my family since.

Preface

"Nature is thrifty in all its actions"
−Maupertuis

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Abstract

In this thesis we study a modern formalism known as Nonequivalent Lagrangian Mechanics, that is constructed on top of the traditional Lagrangian theory of mechanics. By making use of the non-uniqueness of the Lagrangian representation of dynamical systems, we are able to generate conservation laws in a way that is novel and, in many cases much faster than the traditional Noetherian analysis. In every case that we examine, these invariants turn out to be Noetherian invariants in disguise. We apply this theory to a wide variety of systems including predator-prey dynamics and damped driven harmonic motion.

Introduction

The overall project of classical mechanics is to predict the future motions and behaviors of classical systems. Given a particular object with a known set of forces acting on it, as well as appropriate information about the initial position and velocity, we want to predict its position at all later times. Over the last several centuries a large number of methods have been developed to approach this problem.

These expressions come in two flavors, the description of a quantity as it changes across time, often called a trajectory, and the description of a quantity that does not change across time, called a conservation law or an invariant. Each of them give insight into the actual processes involved in the system at hand. In many cases the full knowledge of each of these expressions amounts to a complete description of a system, up to selection of initial conditions.[1]

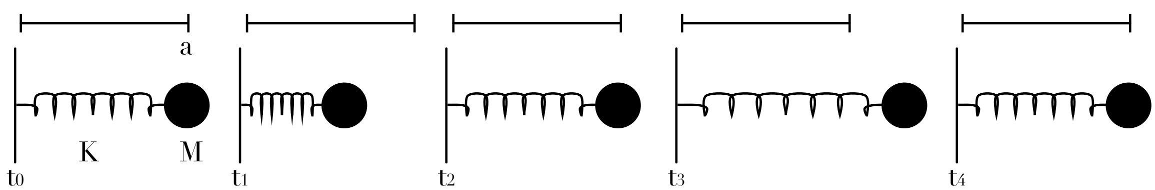

Figure 1: Diagram of the time evolution of a simple harmonic oscillator, with equilibrium length a, spring constant K and mass M.

Consider one of the most basic systems in physics, the simple harmonic oscillator, as depicted in Figure 1. The only force on the mass is given by the usual Hooke's Law restorative force, $F = Ma = -Kx$, which can be rearranged into a differential equation given by

$$\ddot{x} = -\frac{K}{M}x = -\omega_0^2 x\,,$$

where ω_0 is the natural frequency of the oscillator. The solution to this equation, which is given by

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)\,,$$

describes the position of the mass across time, where A and B are fixed by the initial

[1]Systems that do not contain enough invariants to be completely described are known as non-integrable systems.

conditions. The velocity is simply the time-derivative of this expression, and is given by

$$v(t) = -A\omega_0 \sin(\omega_0 t) + B\omega_0 \cos(\omega_0 t)\,.$$

As time passes in the system, velocity and position engage in a tug of war, such that the total mechanical energy of the system

$$E = \frac{1}{2}Mv^2 + \frac{1}{2}Kx^2\,,$$

remains a constant. As can be seen in Figure 2, when the mass gets farther away from the equilibrium the velocity becomes small, while when it is close to the equilibrium the position becomes small and so to compensate the velocity becomes large. There are a great variety of types of conservation laws including conservation of energy, momentum, angular momentum, spin, charge.

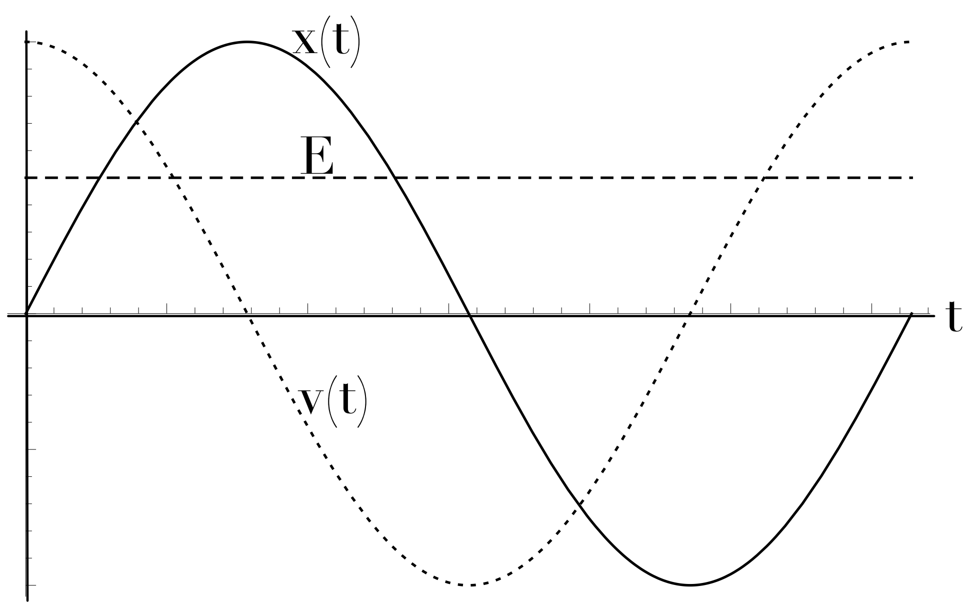

Figure 2: Diagram of the relative behavior of the position, velocity and energy for the simple harmonic oscillator over time.

The modern explanation of why these quantities remain invariant across time is due to Noether's theorem, which says that if a system exhibits a symmetry of a certain type then there exists a corresponding conservation law. In order to fully understand this theorem we must first develop a mature notion of symmetry. While we may be familiar with symmetry in everyday life, be it from art, nature or otherwise, the formal meaning of the concept in physics rests in a system's relationship to transformations. A system has a symmetry if it remains unchanged across a transformation. For instance,

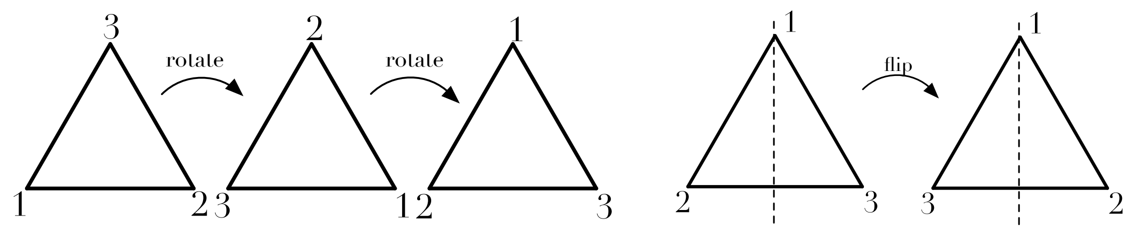

Figure 3: Diagrams of the rotational (left) and axial (right) symmetries of the equilateral triangle.

an equilateral triangle will remain precisely the same under rotations of 60° degrees about its center, as well as flipping about one of its central axes, as in Figure 3. One would say that the equilateral triangle has rotational and axial or mirror symmetry.

This is a particular example of the type of symmetry known as *discrete symmetry*, because each of these transformations is the smallest one that will yield a valid symmetry and yet it is bigger than infinitely small. If we were to rotate the triangle by 1°, or any angle other than a multiple of 60°, then it would appear different than the original.

Conversely, imagine a system that when rotated an infinitesimal amount stays the same. This is true of the circle, as in Figure 4. This type of transformation represents a *continuous symmetry*, because there is a continuous family of rotations that will leave the circle the same.

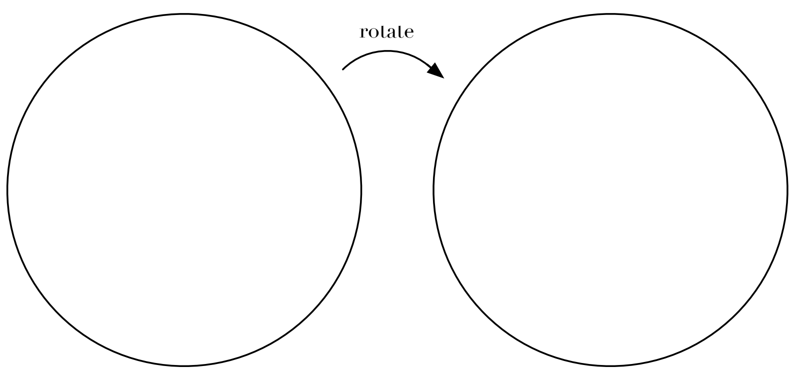

Figure 4: Diagram of a circle under rotational transformation.

Both continuous and discrete symmetries exist in physical systems, but they are often harder to visualize. For instance, consider our simple harmonic oscillator system. If one were to let the system evolve in time, it would still behave in the same oscillatory manner. For all times future and past, the system continues to behave in its original fashion. By Noether's theorem, this symmetry across time shows that the system has conservation of mechanical energy, just as we noted above.

Let's take a step back in order to describe the historical relationship between mechanics and Noether's theorem. Each stage of development in the field of classical mechanics necessitated a new development in formal mathematics. Calculus was created in order to describe the ideas of what is now known as Newtonian Mechanics, which took the view that the universe could be understood as the aggregation of endless cause and effect relationships, between objects and forces acting on them.

As mathematics and physics developed across the 18th century there grew a need for greater degrees of rigor, and so analytical mechanics was born in conjunction with the calculus of variations. These constructions manifested themselves as Lagrangian and Hamiltonian Mechanics, which take the view that physical systems behave in a optimal way across time, or, that nature is thrifty in its actions. These theories form the backbone of the classical mechanics as it is still used in the modern day.

The original construction of Noether's theorem in 1918 [24] takes place on top of the Lagrangian formalism. By applying the theory of Lie Groups to describe continuous symmetries mathematically, Noether was able to rigorously describe the relationship between symmetries and conservation laws.

Similarly in this thesis we will consider a flavor of mechanics that develops a formalism on top of Lagrangian mechanics. This theory was established by a number of authors in the late 1970's and early 1980's known by a variety of names,[2] including Nonequivalent Lagrangian Mechanics.

The driving idea behind this theory is that Noether's theorem takes too limiting a view on the relationship between symmetries and conserved quantities. By relaxing the requirements of her famous theorem we are able to generate conservation laws that are difficult to acquire via traditional Noetherian analysis. While this alteration does not seem to generate any significant new results of its own, it is able to generate Noetherian conservation laws incredibly quickly.

In Chapter 1, we will develop the elements of Lagrangians mechanics that are the underpinning of the machinery we will be working with and give a proof of Noether's theorem. Chapter 2 begins with an introduction to Nonequivalent Lagrangian Mechanics. We will then outline the continuous-transformation approach to this type of mechanics as well as develop a set of related ansatz methods. Finally in Chapter 3 we will demonstrate that this flavor of mechanics gives results that are equivalent to the Noetherian ones in the case of the damped harmonic oscillator and the damped driven harmonic oscillator.

[2]For a full discussion of the multitude of voices present on this topic, see Appendix A.

1

Classical Theory

In this chapter our purpose is to give a general outline of the traditional classical mechanics involved in this thesis. We will begin by introducing the mathematical formalism of the calculus of variations, and from that position we will be able to define Lagrangian Mechanics. We will then introduce enough Lie group theory such that we will be able to rigorously prove Noether's theorem.

1.1 Lagrangian Mechanics

1.1.1 Fermat's Principle

At the heart of nineteenth century classical mechanics is the principle that natural systems always take the most efficient path. This notion is easy to see in the bending of light when it passes from one material to another, as shown in Figure 1.1. The light bending satisfies Snell's Law of refraction, which is

$$n_1 \sin\theta = n_2 \sin\theta' \,.$$

In this equation θ is the angle of incidence, and θ' the angle of refraction. The values n_1 and n_2 are the indices of refraction of the media, which determine the speed at which light passes through the media. This relationship is behind the structure and design of most geometric optical devices such as lenses and prisms.

The traditional approach taken in electrodynamics is to consider the boundary conditions on either side of the interface. In Figure 1.1, the light has a slower speed on the right side of the boundary, but the same frequency, in order to account for

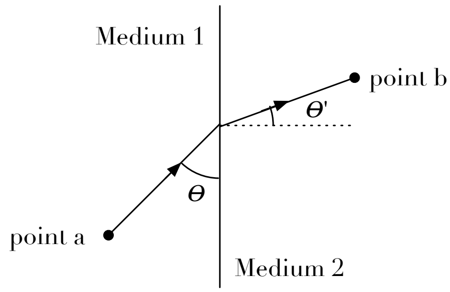

Figure 1.1: Diagram of interface between two medium's whose indexes of refraction cause the described behavior.

this it must bend toward the direction of propagation. This is a cause and effect perspective: the light behaves in one fashion until it reaches an obstruction, at which time it modifies its behavior to account for the hindrance.

However, we can also consider this phenomena from the perspective of Fermat's principle, which says that *light takes the path that minimizes travel time*. Suppose we know that a ray of light starts at point a and ends at point b. While the shortest distance between these points is a straight line, it is not the path that will minimize the travel time, because of the differing speeds of propagation in the media. A careful analysis of what path minimizes travel time reproduces Snell's Law [27].

By viewing the behavior of the system in total, rather than moment by moment, we can make analytical predictions about the behavior of systems as a consequence of optimization across time. This framework provides a fundamentally different way to discuss mechanics than traditional Newtonian or electromagnetic analysis can provide, while still maintaining the results from those fields.

1.1.2 The Principle of Least Action

The notion of extremization leads to Hamilton's famous principle of least action. It states that *all physical systems will act in such a way as to extremize an object known as the action of the system.* Fermat's principle is a particular case of Hamilton's, constructed by letting the action be the total travel time. We generally define the action as

$$S[q(t)] = \int_{t_0}^{t_1} L(q, \dot{q}, t) dt \,, \tag{1.1}$$

where L is a specifically designed function called the Lagrangian. The action is not truly a function, but a functional, which is an entity that maps functions into numbers.

Accordingly the action assigns each trajectory $q(t)$, from t_0 to t_1, a numerical value. While the action is only dependent on the trajectory, the Lagrangian is dependent on $q(t)$, its derivative $\dot{q}(t)$, and time t. The fascinating, and somewhat magical element of this process is that only the true physical trajectory is the one that will cause the action to be minimized.

To extremize the action, we perturb the Lagrangian and find the conditions under which the action integral remains the same.[1] We start by assuming that our particle's trajectory $q(t)$ from point a to point b is action minimizing.

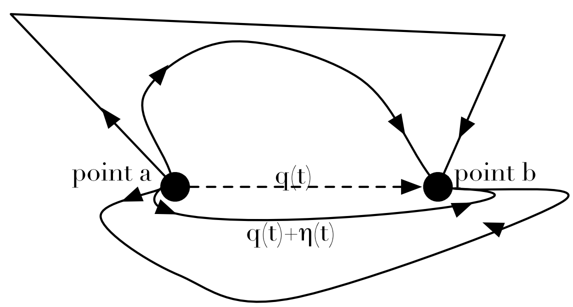

Figure 1.2: Diagram of action minimizing path with a variety of possible perturbations.

We then apply a small perturbation, $\eta(t)$ such that the perturbation vanishes at the end points, but has arbitrary behavior at times between, as in Figure 1.2.[2] In order to adhere to our requirement that the action remain invariant, we require that the change in the value of S be second order in the perturbation:

$$\Delta S = S[q + \eta] - S[q] = \mathcal{O}(\eta^2)\,.$$

Plugging our modified trajectory into the action yields

$$S[q + \eta] = \int_{t_0}^{t_1} L(q + \eta, \dot{q} + \dot{\eta}, t)dt \approx \int_{t_0}^{t_1} \left[L(q, \dot{q}, t) + \frac{\partial L}{\partial q}\eta + \frac{\partial L}{\partial \dot{q}}\dot{\eta} \right] dt\,, \tag{1.2}$$

where we took Taylor expansions to separate the perturbative elements from the original action. We now need to modify one of the components of ΔS, so that it is expressible only in terms of the perturbation, and not its derivative. To do so we will make use of integration by parts:

[1]This derivation closely follows the Extremization of an Action argument in [7].

[2]Despite the apparent randomness and jerks in our diagram the perturbation must be piecewise smooth.[2]

$$\int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{q}} \dot{\eta}\, dt = \eta \frac{\partial L}{\partial \dot{q}} \bigg|_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \eta\, dt = 0 - \int_{t_0}^{t_1} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \eta\, dt\,,$$

Applying this to Equation 1.2 then gives

$$\Delta S = 0 = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial q}\eta - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\eta\right] dt = \int_{t_0}^{t_1} \eta \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] dt\,. \tag{1.3}$$

Recall that η is arbitrary along its path. In general the difference between it and the true path is non-zero, except at the end points, so in order to make $\Delta S = 0$ at all points along the path for any η, we must require that the bracketed term in Equation 1.3 be zero at all points in time. This gives us the famous Euler-Lagrange equation:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} = 0\,.$$

We can generalize our argument so that $\{q_i(t)\}_{i=1}^{N}$ are a set of trajectories for multiple objects or for one object in multiple dimensions. Thus L is a function of the set of trajectories, their velocities, and time, and in turn we obtain a set of Euler-Lagrange equations:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] - \frac{\partial L}{\partial q_i} = 0\,. \tag{1.4}$$

By simply plugging a Lagrangian into Euler-Lagrange equations, we generate a set of second-order differential equations that describe the action-minimized behavior for a given system. Thus the work of mechanics is now in the design and manipulation of Lagrangians.

Traditionally the Lagrangian is given as $L = T - U$: the difference between kinetic and potential energies. This is a useful definition for classical mechanics because it makes the relationship between Lagrangian and Newtonian mechanics quite clear. We recall that kinetic energy in one dimension is typically defined $T = \frac{1}{2}m\dot{x}^2$, and potential energy $U = U(x)$, which gives us the Lagrangian:

$$L = \frac{1}{2}m\dot{x}^2 - U(x).$$

Plugging this into the Euler Lagrange equations from Equation 1.4, we obtain

$$\frac{d}{dt}\left[\frac{2}{2}m\dot{x}\right] - \left(-\frac{\partial U}{\partial x}\right) = m\ddot{x} + \frac{\partial U}{\partial x} = 0 \quad \Rightarrow \quad m\ddot{x} = ma = -\frac{\partial U}{\partial x} = F\,,$$

which is exactly the familiar form of Newton's second law in one dimension.

This prescription allows us to describe the behavior of a vast class of physical systems. Further, it allows for the rapid derivation of the equations of motion in cases that would be frustratingly unwieldy to describe with the language of Newtonian mechanics. However, a Lagrangian defined as the difference between kinetic and potential energy is inapplicable to many mechanical systems. For instance, this recipe cannot accurately describe dissipative motion, or systems under the influence of non-conservative forces, where we cannot construct an appropriate potential energy.

While the prescription $L = T - U$ is extremely useful for the generation of Lagrangians, it is not a rule[10]. A more apt definition of the Lagrangian is that it is a measure of coordinate-space for second-order differential equation systems. This new definition allows us to construct Lagrangians that follow all of the appropriate rules and generate the correct equations of motion, but bear no relationship to the concepts of kinetic and potential energy.

Example: Linearly Damped Motion

Consider the reasonably familiar case of linearly damped motion, which has equation of motion $\ddot{x} = -\beta\dot{x}$. This type of equation occurs naturally in systems such as an object moving through a fluid and experiencing drag. The nontraditional Lagrangian

$$L = \frac{1}{2}\dot{x}^2 e^{\beta t},$$

successfully yields the correct equations of motion:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] - \frac{\partial L}{\partial x} = \frac{d}{dt}\left[\dot{x}e^{\beta t}\right] - 0 = \ddot{x}e^{\beta t} + \beta\dot{x}e^{\beta t} = (\ddot{x} + \beta\dot{x})e^{\beta t} = 0 \quad \Rightarrow \quad \ddot{x} = -\dot{x}\beta\,.$$

This suggests that the Lagrangian is a mathematical object, not a physical one. While we are able to use it in a number of ways to gain physical information about dynamical systems, such as the trajectory of particles in the system or conservation laws that are in action, the Lagrangian *itself* has no physics in it.

We take this opportunity to point out that Lagrangians are not unique. We can easily write down a second Lagrangian that will give us linearly damped motion, such as

$$\tilde{L} = \dot{x}\ln\dot{x} - \beta x.$$

We can demonstrate that this Lagrangian also yields the desired equations of motion:

$$\frac{d}{dt}\left[\frac{\partial \tilde{L}}{\partial \dot{x}}\right] - \frac{\partial \tilde{L}}{\partial x} = \frac{d}{dt}\left[\ln\dot{x} + 1\right] + \beta = \frac{\ddot{x}}{\dot{x}} + \beta = 0 \quad \Rightarrow \quad \ddot{x} = -\dot{x}\beta\,.$$

Both of our Lagrangians are valid, but don't follow the traditional recipe for writing Lagrangians. While there are stories one could tell oneself about the physical meaning of these nontraditional Lagrangians, such as the interpretation that L describes a free particle with exponentially increasing mass, it is important to remember that they are not necessarily physically meaningful.

1.2 Symmetry

Symmetry arises in this form of classical mechanics by way of coordinate transformations. A continuous transformation that leaves the action invariant is known as a Noether symmetry. We wish to present a derivation of the relationship between this type of transformation and conserved quantities.

1.2.1 Coordinate Transformations

Continuous Symmetry transformations in classical mechanics can be described in Lie Group Theory. A Lie group is a group that acts on a smooth differentiable manifold and describes a continuous symmetry.[3] In this case the manifold we are considering is the collection of possible trajectories that satisfies the equations of motion, defined by

$$\alpha(q, \dot{q}, t) = \ddot{q},$$

and derived from an Euler-Lagrange equation. We start by considering an infinitesimal transformation in space and time from the arbitrary variables q and t to Q and T. We introduce functions η and τ and use them to perturb our original coordinates:

$$Q = q + \epsilon\eta(q, \dot{q}, t)\,, \qquad\qquad T = t + \epsilon\tau(q, \dot{q}, t)\,, \tag{1.5}$$

where ϵ is a constant known as the group parameter, which we take to be small. We can see that these transformations form a group, as associativity is explicitly built into the real number system, and the inverse and identity are simply $(\tau \to -\tau, \eta \to -\eta)$ and $(\tau \to 0, \eta \to 0)$ respectively.

We also need to know how $\dot{q}$ changes under these transformations. To do so we simply modify our transformed velocity so that we can use Equation 1.5:

$$Q' = \frac{dQ}{dT} = \frac{dQ/dt}{dT/dt} = \frac{\dot{q} + \epsilon\dot{\eta}}{1 + \epsilon\dot{\tau}} \approx (\dot{q} + \epsilon\dot{\eta})(1 + \epsilon\dot{\tau}) = \dot{q} + \epsilon(\dot{\eta} - \dot{\tau}\dot{q})\,,$$

where we have dropped the second order terms.[4] We can package these three terms

[3]Because the scope of this language goes far beyond the needs of this thesis, we will simply make use of a watered down version that is primarily concerned with one class of transformations. We refer the reader to [9] for a lively discussion of the full version of theory as it relates to physics.

[4]In this section we use the prime to denote total derivatives with respect to the upper-case variable

together into an object known as an infinitesimal generator:

$$E = \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial q} + (\dot{\eta} - \dot{\tau}\dot{q})\frac{\partial}{\partial \dot{q}} \,. \tag{1.6}$$

By applying this object to a function, its coordinates are modified in the fashion prescribed by the generator. The action of the generator on our basic coordinates is

$$\begin{aligned} Q &= e^{\epsilon E} q \approx (1+\epsilon E)q = q + \epsilon\eta \,, \\ T &= e^{\epsilon E} t \approx (1+\epsilon E)t = t + \epsilon\tau \,, \\ \dot{Q} &= e^{\epsilon E}\dot{q} \approx (1+\epsilon E)\dot{q} = \dot{q} + \epsilon(\dot{\eta} - \dot{\tau}\dot{q}) \,. \end{aligned} \tag{1.7}$$

Of course if we set ϵ to a finite value, then the application of the generator describes a collection of finite transformations. This construction has been completely general and is applicable to all forms of continuous transformations. They are made useful by imposing constraints on η and τ, relevant to the system at hand. Thus the work of using this type of machinery is finding the functions of η and τ that constitute a valid symmetry.

1.2.2 Noether's Theorem

We are now in a position to formally prove Noether's theorem, which entails a transformation that causes the action to remain invariant. In order to do so, we must find the transformation that leaves the Lagrangian invariant up to an additive total time derivative, ϵJ. To that end we introduce the transformations from Equation 1.7 into the a generic Lagrangian L:

$$\begin{aligned} &\left(L + \epsilon\frac{dJ}{dt}\right)dt = L(Q, Q', T)dT \quad \Rightarrow \quad L + \epsilon\frac{dJ}{dt} = L(Q, Q', T)\frac{dT}{dt} \\ \Rightarrow \quad &\tilde{L}(Q, Q', T) = L(Q, Q', T)(1 + \epsilon\dot{\tau}) \,. \end{aligned}$$

If we explicitly plug our transformations into the Lagrangian,

$$\tilde{L}(Q, Q', T) = L\Big(q + \eta\epsilon, \dot{q} + \epsilon(\dot{\eta} - \dot{q}\dot{\tau}), t + \epsilon\tau\Big)\left(1 + \dot{\tau}\epsilon\right) \,,$$

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and dot to denote total derivatives with respect to lower case, i.e.

$$Q' = \frac{dQ}{dT} \,, \qquad \dot{Q} = \frac{dQ}{dt} \,.$$

and take the Taylor expansion up to first order, we obtain

$$\begin{aligned} L(Q,Q',T) &= L(q,\dot{q},t) \\ &+ \epsilon\left[\dot{\tau}L(q,\dot{q},t) + \eta\frac{\partial L(q,\dot{q},t)}{\partial q} + (\dot{\eta} - \dot{q}\dot{\tau})\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}} + \tau\frac{\partial L(q,\dot{q},t)}{\partial t}\right] . \end{aligned} \tag{1.8}$$

By making use of the first and second forms of the Euler-Lagrange Equations[5] we can reduce the bracketed term in Equation 1.8 to

$$\eta\frac{\partial L}{\partial q} + \dot{\eta}\frac{\partial L}{\partial \dot{q}} + \tau\frac{\partial L}{\partial t} + \dot{\tau}\left(L - \dot{q}\frac{\partial L}{\partial \dot{q}}\right) = \frac{d}{dt}\left[\eta\frac{\partial L}{\partial \dot{q}} + \tau\left(L - \dot{q}\frac{\partial L}{\partial \dot{q}}\right)\right] .$$

We plug this into Equation 1.8, which gives

$$L(Q,\dot{Q},T) = L(q,\dot{q},t) + \epsilon\frac{d}{dt}\left[\eta\frac{\partial L}{\partial \dot{q}} + \tau\left(L - \dot{q}\frac{\partial L}{\partial \dot{q}}\right)\right] = L(q,\dot{q},t) + \epsilon\frac{dJ}{dt} .$$

If we equate the two terms under the total time derivatives,

$$\frac{dJ}{dt} = \frac{d}{dt}\left[\eta\frac{\partial L}{\partial \dot{q}} + \tau\left(L - \dot{q}\frac{\partial L}{\partial \dot{q}}\right)\right] ,$$

this implies that the quantity

$$\Phi_N = \eta\frac{\partial L}{\partial \dot{q}} + \tau\left(L - \dot{q}\frac{\partial L}{\partial \dot{q}}\right) - J ,$$

is invariant. Just as in the case of the Euler-Lagrange equations, we can generalize this argument to a system with multiple dynamical variables. Having assembled all of the necessary pieces, we can now fully state Noether's theorem:

[5]Using the Euler-Lagrange equation we can derive the second form of the Euler-Lagrange equation, which reads:

$$\frac{\partial L}{\partial t} = \frac{dL}{dt} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\dot{q}\right] .$$

Noether's Theorem

For a given physical system, a continuous transformation of the form

$$q_i \to q_i + \epsilon\eta_i(q_j, \dot{q}_j, t)\,, \qquad\qquad t \to t + \epsilon\tau(q_j, \dot{q}_j, t)\,, \tag{1.9}$$

that satisfies the Killing equation, which is given by

$$\dot{\tau}L(q_i, \dot{q}_i, t) + \eta\frac{\partial L(q_i, \dot{q}_i, t)}{\partial q_i} + (\dot{\eta}_i - \dot{q}_i\dot{\tau})\frac{\partial L(q_i, \dot{q}_i, t)}{\partial \dot{q}_i} + \tau\frac{\partial L(q_i, \dot{q}_i, t)}{\partial t} = \frac{dJ}{dt}\,, \tag{1.10}$$

will generate an invariant of the form

$$\Phi_N = \eta_i\frac{\partial L}{\partial \dot{q}_i} + \tau\left(L - \dot{q}_i\frac{\partial L}{\partial \dot{q}_i}\right) - J\,, \tag{1.11}$$

where the repeated indices are summed over.

We will conclude this chapter with a pair of examples that demonstrate the easy power of this famous theorem.

Example: Translational Motion

Consider a generic first system, the free particle in one dimension. From our prior knowledge of the system, linear momentum should be conserved. Our Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2\,.$$

Clearly this will have the usual equation of motion, $\ddot{x} = 0$. We use an infinitesimal translational transformation by setting $\eta = 1$ and $\tau = 0$, which yield transformations

$$x = x + \epsilon, \quad t = t + 0\epsilon = t, \quad \dot{x} = \dot{x}.$$

This gives the exact same Lagrangian, and so $J = 0$. We use this to derive our Noether Invariant:

$$\Phi_N = \eta\frac{\partial L}{\partial \dot{q}} + \tau\left(L - \dot{q}\frac{\partial L}{\partial \dot{x}}\right) - J = m\dot{x} - 0 = m\dot{x}\,.$$

Clearly momentum is conserved, just as we suspected it would be.

Let's take another look at this example with a twist. We consider a new Lagrangian:

$$L = \frac{1}{2}m\dot{x}^2 + \dot{x}x\,.$$

Our new Lagrangian is different from the first just by a total time derivative, namely

$$\frac{d}{dt}\left(\frac{1}{2}x^2\right) = x\dot{x}\,.$$

This shouldn't affect the Euler-Lagrange equations, which we can check by writing

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \frac{d}{dt}\left(m\dot{x} + x\right) - \dot{x} = m\ddot{x} + \dot{x} - \dot{x} = m\ddot{x} = 0\,.$$

The same transformation we used above will also work for this Lagrangian:

$$\tilde{L} = \frac{1}{2}m\dot{x}^2 + \dot{x}(x+\epsilon) = \frac{1}{2}m\dot{x}^2 + \dot{x}x + \epsilon\dot{x} = L + \epsilon\frac{d}{dt}(x)\,.$$

Thus $J = x$, and we can now write down the invariant associated with this transformation:

$$\Phi_N = \eta\frac{\partial L}{\partial \dot{x}} - J = (m\dot{x} + x) - x = m\dot{x}\,.$$

Clearly this system maintains its invariance across linear spatial transformation.

Example: Conserved Hamiltonian

A commonly known fact is that the Hamiltonian of a system is conserved when its associated Lagrangian has time-translation symmetry. Consider an infinitesimal temporal transformation, such that $\eta = 0$ and $\tau = 1$. This yields transformations:

$$x = x + 0\epsilon = x\,, \quad t = t + \epsilon\,, \quad \dot{x} = \dot{x}\,,$$

which we apply to a generic Lagrangian L, and obtain

$$\begin{aligned} L &= L(X_i, \dot{X}_i, T)\frac{dt}{dT} = L(x_i, \dot{x}_i, t+\epsilon)(1+\dot{\tau}\epsilon) = \left(L(x_i, \dot{x}_i, t) + \frac{\partial L}{\partial t}\epsilon\right)(1+0) \\ &= L + \frac{\partial L}{\partial t}\epsilon \quad \Rightarrow \quad \frac{dJ}{dt}\epsilon = \frac{\partial L}{\partial t}\epsilon \quad \Rightarrow \quad J = \int \frac{\partial L}{\partial t} dt\,. \end{aligned}$$

We now construct the associated invariant:

$$\Phi_N = 0\frac{\partial L}{\partial \dot{x}_i} + 1\left(L - \dot{x}_i\frac{\partial L}{\partial \dot{x}_i}\right) - \left(\int \frac{\partial L}{\partial t} dt\right) = \left(L - \dot{x}_i\frac{\partial L}{\partial \dot{x}_i}\right) - \left(\int \frac{\partial L}{\partial t} dt\right)\,.$$

In the case where the Lagrangian has no explicit time dependence, that is when $\partial L/\partial t = 0$, then we have

$$\Phi_N = L - \dot{q}_i\frac{\partial L}{\partial \dot{x}_i} = -H\,,$$

as a conserved quantity. This is of course exactly the definition of the Hamiltonian modulo a minus sign.

2

Nonequivalent Lagrangian Mechanics

The central requirement of Noether's theorem, that the action remain invariant under transformation, appears to restrict the types of symmetries that are allowed. If this requirement is relaxed in such a way that the behavior of the action doesn't matter but the equations of motion that are generated remain the same, a large number of conservation laws in unusual forms become available for study.

If you begin with one Lagrangian and transform it in a way that leaves the equations of motion invariant, you typically do not leave the action invariant. Thus, you obtain a second Lagrangian that is different from the first by more than a total time derivative. Using this pair of nonequivalent Lagrangians, we will be able to build a conserved quantity.

2.1 The Lutzky Invariant

Following the work of M. Lutzky [17, 18, 19, 20] we will consider a pair of Lagrangians, L and $\tilde{L}$, for a given physical system with equations of motion

$$\ddot{q}_i = \alpha_i(q_j, \dot{q}_j, t).$$

If the Lagrangians are different by more than a total time derivative, we refer to them as Nonequivalent. Each of these Lagrangians has a corresponding value known as Jacobi's Last Multiplier [23], which is defined:

$$\tilde{M} = \det\left(\frac{\partial^2 \tilde{L}}{\partial \dot{q}_i \partial \dot{q}_j}\right), \qquad M = \det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right),$$

where the argument of the determinant is a symmetric matrix known as the Hessian of the Lagrangian:

$$\mathbb{M}_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \,.$$

With these tools in hand we can define the central piece of this theory, the Lutzky Invariant, which is given by

$$\Phi_L = \frac{\tilde{M}}{M} \,.$$

We can show that the total time derivative of this quantity is zero, $\dot{\Phi}_L = 0$, provided the Euler Lagrange equations are satisfied and the equations of motion are shared between the Lagrangians. We begin by taking the logarithm of the JLM, and making use of a common matrix identity:

$$\ln\big(\det \mathbb{M}\big) = \mathrm{Tr}\big(\ln \mathbb{M}\big) \,.$$

If we take the total derivative we obtain

$$\begin{aligned}\frac{d}{dt}\mathrm{Tr}\ln\left[\mathbb{M}\right] &= \mathbb{M}^{-1}_{ij}\frac{d\mathbb{M}_{ij}}{dt} = \mathbb{M}^{-1}_{ij}\frac{d}{dt}\left[\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right] \\ &= \mathbb{M}^{-1}_{ij}\left[\frac{\partial^3 L}{\partial t \partial \dot{q}_i \partial \dot{q}_j} + \frac{\partial^3 L}{\partial \dot{q}_i \partial \dot{q}_j \partial q_k}\dot{q}_k + \frac{\partial^3 L}{\partial \dot{q}_i \partial \dot{q}_j \partial \dot{q}_k}\ddot{q}_k\right] \\ &= \mathbb{M}^{-1}_{ij}\left[\frac{\partial}{\partial \dot{q}_i}\left(\frac{\partial^2 L}{\partial t \partial \dot{q}_j} + \frac{\partial^2 L}{\partial \dot{q}_j \partial q_k}\dot{q}_k\right) - \frac{\partial}{\partial \dot{q}_j}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial^2 L}{\partial \dot{q}_i \partial q_k}\ddot{q}_k\right)\right. \\ &\qquad \left. - \frac{\partial \ddot{q}_k}{\partial \dot{q}_j}\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_k}\right] \,.\end{aligned}$$

We note that the equations of motion are defined by $\ddot{q}_k = \alpha_k$, so this is

$$= \mathbb{M}^{-1}_{ij}\left[\frac{\partial}{\partial \dot{q}_i}\left(\frac{\partial^2 L}{\partial t \partial \dot{q}_j} + \frac{\partial^2 L}{\partial \dot{q}_j \partial q_k}\dot{q}_k\right) - \frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial t} + \frac{\partial^2 L}{\partial \dot{q}_i \partial q_k}\dot{q}_k\right) - \frac{\partial \alpha_k}{\partial \dot{q}_j}\mathbb{M}_{ik}\right] \,.$$

Finally we make use of the fact that $\mathbb{M}_{ij}$ is symmetric, so the first two terms in the bracket above cancel, leaving

$$\mathbb{M}^{-1}_{ij}\frac{\partial \alpha_k}{\partial \dot{q}_j}\mathbb{M}_{ik} = -\delta_{jk}\frac{\partial \alpha_k}{\partial \dot{q}_j} = -\frac{\partial \alpha_k}{\partial \dot{q}_k} = \frac{d}{dt}\mathrm{Tr}\ln\mathbb{M} \,.$$

That is, we have demonstrated that

$$\frac{d}{dt}\ln M = -\frac{\partial \alpha_k}{\partial \dot{q}_k}\,.$$

This is true for all second-order Lagrangians that are non-singular, (Lagrangians whose corresponding Jacobi Last Multiplier is non-zero). Naturally all of this machinery also holds for the second Lagrangian, $\tilde{L}$. We can therefore form the corresponding statement for $\tilde{M}$ and subtract one from the other:

$$\frac{d}{dt}\left(\ln \tilde{M}\right) - \frac{d}{dt}\left(\ln M\right) = \frac{d}{dt}\left(\ln\left[\frac{\tilde{M}}{M}\right]\right) = -\frac{\partial \alpha_i}{\partial \dot{q}_i} - \left(-\frac{\partial \alpha_i}{\partial \dot{q}_i}\right) = 0\,.$$

Thus at all points in time the ratio of the Jacobi Last Multipliers, Φ_L, will be constant.

Lutzky Invariant

For a given physical system that is expressible in terms of two Lagrangians, and whose Jacobi Last Multipliers are non-zero, the ratio

$$\Phi_L = \frac{\tilde{M}}{M} = \det\left(\frac{\partial^2 \tilde{L}}{\partial \dot{q}_i \partial \dot{q}_j}\right)\det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right)^{-1}, \tag{2.1}$$

will be an invariant of the motion.

While this quantity has been addressed by a variety of authors [6, 19, 20, 11], this specific version of this construction is due to Lutzky, and so we credit him. For additional discussion of this matter, see Appendix A.

Example: Linearly Damped Motion

As a first example recall our pair of Lagrangians from our the previous chapter that lead to the equations of motion for linear damped motion, which are $\ddot{x} = -\beta\dot{x}$:

$$\tilde{L} = \frac{1}{2}\dot{x}^2 e^{\beta t}\,, \qquad\qquad L = \dot{x}\ln\dot{x} - \beta x\,.$$

We can construct the Lutzky invariant from these quantities:

$$\Phi_L = \left(\frac{\partial^2 \tilde{L}}{\partial \dot{x}^2}\right)\left(\frac{\partial^2 L}{\partial \dot{x}^2}\right)^{-1} = \frac{e^{t\beta}}{1/\dot{x}} = \dot{x}e^{t\beta}\,.$$

Fortunately for us we can also solve the equations of motion analytically:

$$\ddot{x} = -\beta\dot{x} \quad\Rightarrow\quad \dot{g} = -\beta g \quad\Rightarrow\quad g = \dot{x} = Ae^{-t\beta} \quad\Rightarrow\quad x = -\frac{A}{\beta}e^{-t\beta} + c_1\,,$$

and when we plug this into our formula for Φ_L, we obtain

$$\Phi_L = Ae^{-t\beta}e^{t\beta} = A\,,$$

which is perfectly invariant.

Example: Free Particle

As a second example consider the system of a free particle moving in a Newtonian gravitation field. The equations of motion are $\ddot{x} = 0$ and $\ddot{y} = -g$. For this system we can construct two Lagrangians

$$L = \frac{1}{2}\dot{y}^2 + gy + \frac{1}{2}\dot{x}^2\,, \qquad \tilde{L} = c_1 e^{\dot{x}} + c_2 e^{-\dot{y}^2/2 - gy}\left(2 + e^{\dot{y}^2/2}\sqrt{2\pi}\dot{y}\ \mathrm{Erf}\left[\frac{\dot{y}}{\sqrt{2}}\right]\right)\,.$$

The first Lagrangian here is the traditional $L = T - U$. The second is more obscure and was found using methods that we will develop in Section 2.3. We will now form the corresponding Hessians and their corresponding Jacobi Last Multipliers:

$$\mathbb{M} = \frac{\partial \mathcal{L}}{\partial \dot{x}_i \partial \dot{x}_j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \Rightarrow \qquad M = 1,$$

$$\tilde{\mathbb{M}} = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{x}_i \partial \dot{x}_j} = \begin{pmatrix} c_1 e^{\dot{x}} & 0 \\ 0 & c_2 e^{-\frac{1}{2}\dot{y}^2 - gy} \end{pmatrix} \qquad \Rightarrow \qquad \tilde{M} = c_1 c_2 e^{\dot{x} - \frac{1}{2}\dot{y}^2 - gy}.$$

Defining $c_3 = c_1 c_2$, the corresponding Lutzky Invariant is simply:

$$\Phi_L = \frac{\tilde{M}}{M} = c_3 e^{\dot{x} - \frac{1}{2}\dot{y}^2 + gy}\,.$$

If we differentiate this with respect to time we obtain

$$\frac{d\Phi}{dt} = c_3 e^{\dot{x} - \frac{1}{2}\dot{y}^2 + gy}\left(\ddot{x} - g\dot{y} - \frac{2}{2}\dot{y}\ddot{y}\right) = c_3 e^{\dot{x} - \frac{1}{2}\dot{y}^2 - gy}\left(\ddot{x} - \dot{y}(g + \ddot{y})\right) = 0\,.$$

2.2 Lutzky Symmetry Transformations

Lutzky approached the problem of generating nonequivalent Lagrangians by making use of the Lie transformation language.[1] We begin by declaring that we are interested in the class of symmetries that leave the equations of motion invariant, rather than the action. We can express this requirement symbolically by developing a generator that incorporates acceleration transformations. That is, we need to know how $\ddot{q}$ transforms under the transformations

$$t \to t + \epsilon\tau\,, \qquad q_i \to q_i + \epsilon\eta_i\,, \qquad \dot{q}_i \to \dot{q}_i + \epsilon(\dot{\eta}_i - \dot{q}_i\dot{\tau})\,.$$

We can write

$$Q_i'' = \frac{dQ_i'}{dT} = \frac{dQ_i'/dt}{dT/dt} = \frac{\ddot{q}_i + \epsilon(\ddot{\eta}_i - \ddot{\tau}\dot{q}_i - \dot{\tau}\ddot{q}_i)}{1+\epsilon\dot{\tau}} \approx \ddot{q}_i + \epsilon(\ddot{\eta}_i - \ddot{\tau}\dot{q}_i - 2\ddot{q}_i\dot{\tau})\,.$$

Thus, we see that the new generator is given by:

$$F = \tau\frac{\partial}{\partial t} + \eta_i\frac{\partial}{\partial q_i} + (\dot{\eta}_i - \dot{\tau}\dot{q}_i)\frac{\partial}{\partial \dot{q}_i} + (\ddot{\eta}_i - 2\ddot{\tau}\dot{q}_i - \alpha_i\dot{\tau})\frac{\partial}{\partial \ddot{q}_i}\,.$$

We now let the generator act on the equations of motion, $\ddot{q}_i = \alpha_i(q_i, \dot{q}_i, t)$. Because α does not have $\ddot{q}$-dependence, we can say that this transformation maintains the equations of motion if $F(\ddot{q}) = E(\alpha)$, where E is the generator from Equation 1.6. This gives

$$F(\ddot{q}_i) = 0 + 0 + 0 + (\ddot{\eta}_i - 2\ddot{\tau}\dot{q}_i - \alpha_i\dot{\tau})\frac{\partial \ddot{q}_i}{\partial \ddot{q}_i} = \ddot{\eta}_i - \ddot{\tau}\dot{q}_i - 2\alpha_i\dot{\tau} = E(\alpha_i)\,.$$

Thus the equation that constrains which transformations are allowed for a particular system is given by

$$\ddot{\eta}_i - \dot{q}_i\ddot{\tau} - 2\dot{\tau}\alpha_i - E(\alpha_i) = 0\,. \tag{2.2}$$

This is known as the determining equation for Lie transformations. If this equation is satisfied by a given η_i and τ, then the equations of motion are symmetric across this transformation.

[1] This derivation closely follows the techniques developed in [17] and the more refined derivation found in [13].

Example: Free Particle

The most elementary example that we can use here is the free particle. As always it has equations of motion $\ddot{x} = \alpha = 0$. We note the group action on α is

$$E(\alpha) = \tau\frac{\partial\alpha}{\partial t} + \eta\frac{\partial\alpha}{\partial x} + (\dot{\eta} - \dot{\tau}\dot{x})\frac{\partial\alpha}{\partial\dot{x}} = \tau 0 + \eta 0 + (\dot{\eta} - \dot{\tau}\dot{x})0 = 0\,,$$

and therefore the determining equation is

$$\ddot{\eta} - \dot{x}\ddot{\tau} - 2\dot{\tau}\alpha - E(\alpha) = \ddot{\eta} - \dot{x}\ddot{\tau} - 2\dot{\tau}0 - 0 = 0 \qquad \Rightarrow \qquad \ddot{\eta} - \dot{x}\ddot{\tau} = 0\,.$$

This appears to be a relatively simple equation, though η and τ are both functions of x, $\dot{x}$ and t. While it is straightforward to generate particular solutions, it is a non-trivial task to completely solve it: the high degree of symmetry that is built into the system leads to a large number of solutions. We elect to guess a simple solution $\tau = x$, $\eta = x$. This gives

$$\ddot{\eta} - \dot{x}\ddot{\tau} = \ddot{x} - \dot{x}\ddot{x} = 0 - 0 = 0\,,$$

so our transformations satisfies the determining equation. We now implement the infinitesimal transformations:

$$X = x + \epsilon x\,, \qquad T = t + \epsilon x\,, \qquad \dot{X} = \dot{x}\left(1 - \epsilon(1 - \dot{x})\right)\,.$$

The canonical Lagrangian for this situation is $L = \frac{1}{2}\dot{x}^2$, (setting the mass of the particle to $m = 1$), which becomes

$$\tilde{L} = \frac{1}{2}(\dot{X})^2\left(\frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\dot{X}\right) = \frac{1}{2}\dot{x}^2(1 - 2\epsilon - \dot{x}\epsilon)\,,$$

under transformation. We can then check to ensure that equations of motion are correct:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial\tilde{L}}{\partial\dot{x}}\right) - \frac{\partial\tilde{L}}{\partial x} &= \frac{d}{dt}\left(\dot{x}(1 - 2\epsilon - \dot{x}\epsilon) - \frac{1}{2}\dot{x}^2\epsilon\right) \\ &= -2\epsilon\dot{x}\ddot{x} + (1 + 2\epsilon - \epsilon\dot{x})\ddot{x} = \ddot{x}(1 + 2\epsilon - 2\epsilon\dot{x}) = 0 \\ &\Rightarrow \qquad \ddot{x} = 0\,. \end{aligned}$$

Finally we can compute the Lutzky invariant:

$$\Phi_L = \left(\frac{\partial^2 \tilde{L}}{\partial \dot{x}^2}\right)\left(\frac{\partial^2 L}{\partial \dot{x}^2}\right)^{-1} = \left((1-2\epsilon-\dot{x}\epsilon)-\dot{x}\epsilon-\dot{x}\epsilon\right)(1)^{-1} = 1-\epsilon(2+3\dot{x})\,.$$

This quantity is invariant because the particle momentum $\dot{x}$ is conserved.

2.2.1 Possible Issues

In a situation when the transformations η_i and τ are only dependent on q_i and t, the Lutzky invariant works coherently and gives the results that we have designed it to give. However, possible issues arise when the transformations depend on $\dot{q}_i$.

In this or any other situation, solutions to the determining equation uniformly transform the Lagrangian into a valid non-equivalent one. However, when the transformations are velocity-dependent computation of the equations of motion requires a little extra care. In this case, the transformed Lagrangians will generally be dependent on $\ddot{q}_i$ in addition to the standard t, q, and $\dot{q}$. To derive the equations of motion, we need to make use of Euler-Lagrange equations for Lagrangians that are dependent on acceleration,[2] which are given by

$$\frac{d}{dt}\left(\frac{d}{dt}\left[\frac{\partial L}{\partial \ddot{q}_i}\right]\right) - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] + \frac{\partial L}{\partial q_i} = 0\,.$$

It is somewhat unnerving to be making use of the third order form of the Euler-Lagrange equation in the context of the formalism built up around the physical manifold defined by the equations of motion that are second order in time. However, because of the way we have constructed the determining equations, the third-order terms always perfectly cancel in the process of constructing the equations of motion.

Throughout his work on this topic, Lutzky makes use of the fact that $\ddot{q}_i = \alpha(q_i, \dot{q}_i, t)$, and un-discriminatingly uses this relationship to flip back and forth between representations. While this does allow his transformations to work on the equations of motion, it will frequently generate Lagrangians that do not yield the right equations of motion or generate a valid invariant.

While demanding that the equations of motion cannot be used in the construction of new Lagrangians, and making use of the third order form of the Euler-Lagrange

[2] The derivation of these equations is identical to the one given in Section 1.1.2 with one additional integration by parts argument, and so we can simply elect to quote them.

equations will cause the transformed Lagrangians to give the right equations of motion, our construction in Section 2.1 of the invariant assumes both Lagrangians are only dependent on $q_i, \dot{q}_i$, and t. If we add dependence on $\ddot{q}_i$, the proof no longer works and we do not generally get a valid Lutzky invariant. So we must add a new constraint in order to guarantee the transformed Lagrangian, $\tilde{L}$, both successfully gives the correct equations of motion and yields a valid Lutzky invariant, which is simply that $\frac{\partial \tilde{L}}{\partial \ddot{q}} = 0$. We can state this in the form of a theorem:

Infinitesimal Transformations and the Lutzky Invariant

For a given physical system with equations of motion $\ddot{q}_i = \alpha_i(q_j, \dot{q}_j, t)$, *transformations that satisfy the equation*

$$\ddot{\eta}_i - \dot{q}_i\ddot{\tau} - 2\dot{\tau}\alpha_i - E(\alpha_i) = 0\,, \tag{2.3}$$

will generate a valid Nonequivlant Lagrangian. However, if the transformed Lagrangian is explicitly dependent on acceleration, that is if

$$\frac{\partial \tilde{L}}{\partial \ddot{q}} = 0\,, \tag{2.4}$$

then that Lagrangian will not lead to a valid Lutzky Invariant.

Example: Linear Dampened Motion

We return to our usual example, the Linear Dampened Motion system. As always, it has equation of motion $\ddot{x} = \alpha = -\beta\dot{x}$. We will start with the nontraditional Lagrangian, $L = \frac{1}{2}\dot{x}^2 e^{\beta t}$.

The determining equation for this system is

$$E(\alpha) = -\beta(\dot{\eta} - \dot{x}\dot{\tau}) \quad \Rightarrow \quad \ddot{\eta} + \beta\dot{\eta} + \dot{x}(\beta\dot{\tau} - \ddot{\tau}) = 0\,.$$

We can construct a solution $\tau = \beta x, \eta = \beta x\dot{x}$, which we can demonstrate is a valid transformation

$$\begin{aligned}\beta(2\dot{x}\ddot{x} + \dot{x}\ddot{x} + x\dddot{x}) + \beta^2(\dot{x}^2 + x\ddot{x}) + \dot{x}(\beta^2\dot{x} - \beta\ddot{x}) &= \\ (-3\beta^2\dot{x}^2 + \beta^3 x\dot{x}) + \beta^2(\dot{x}^2 - \beta x\dot{x}) + (\beta^2\dot{x}^2 + \beta^2\dot{x}^2) &= \\ -3\beta^2\dot{x}^2 + 3\beta^2\dot{x}^2 + \beta^3 x\dot{x} - \beta^3 x\dot{x} &= 0\,.\end{aligned}$$

This gives the transformations,

$$T = t + \epsilon\beta x, \quad X = x + \epsilon\beta x\dot{x}, \quad \dot{X} = \dot{x} + \beta\epsilon x\ddot{x}\,,$$

and the transformed Lagrangian,

$$\tilde{L} = \frac{1}{2}(\dot{x} + \beta\epsilon x\ddot{x})^2 e^{\beta(t+\epsilon\beta x)}(1 + \epsilon\beta\dot{x}) = \frac{1}{2}\dot{x}^2 e^{\beta t} + \frac{1}{2}\epsilon\beta e^{\beta t}(\dot{x}^3 + \beta x\dot{x}^2 + 2x\dot{x}\ddot{x})\,.$$

We can see clearly that this transformed Lagrangian will not be handled very well by the Lutzky invariant due to its $\ddot{x}$ dependence.

However the equations of motion generated by this Lagrangian are correct:

$$\frac{\partial \overline{L}}{\partial x} = 0 + \frac{1}{2}\epsilon\beta e^{\beta t}(\beta\dot{x}^2 + 2\dot{x}\ddot{x}),$$

$$-\frac{d}{dt}\left[\frac{\partial \overline{L}}{\partial \dot{x}}\right] = \left[\ddot{x}e^{\beta t} + \beta\dot{x}e^{\beta t}\right] + \frac{1}{2}\epsilon\beta e^{t\beta}(\dot{x}(5\beta\dot{x} + 8\ddot{x}) + x(2\beta^2\dot{x} + 4\beta\ddot{x} + 2\dddot{x}))\,,$$

$$+\frac{d}{dt}\left[\frac{d}{dt}\left[\frac{\partial \overline{L}}{\partial \ddot{x}}\right]\right] = \frac{1}{2}\epsilon\beta e^{t\beta}(\dot{x}(4\beta\dot{x} + 6\ddot{x}) + x(2\beta^2\dot{x} + 4\beta\ddot{x} + 2\dddot{x}))\,.$$

Clearly the Euler-Lagrange equations prevent the ϵ terms from contributing to the equations of motion and we get back precisely the ones that we would expect.

2.3 Ansatz Methods

The process of generating nonequivalent Lagrangians using symmetry transformations is non-trivial, and in some cases, due to the difficulty of solving the determining equation, it is prohibitively difficult. As with many other problems in physics, our solution is to use ansatz methods. This means guessing the basic structure of a Lagrangian, applying the Euler-Lagrange equations, and matching the result to the desired equations of motion.

These tools allow us to solve the reasonably broad class of systems whose equations of motion are multiplicatively separable, that is, systems that are given by the following form[3]

$$\ddot{x} = H(x)G(\dot{x}) = F'(x)G(\dot{x}), \qquad \text{where} \qquad F'(x) = H(x).$$

Many systems of mild to great physical interest are constructible in this form, ranging from the simple harmonic oscillator to relativistic motion in a variety of cases. [16] We now need to find an appropriate Lagrangian to generate this equation of motion. We first consider an additive ansatz:

$$L = f(x) + g(\dot{x})\,.$$

This gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \quad \Rightarrow \quad \frac{d}{dt}(g') = f' \quad \Rightarrow \quad g''\ddot{x} = f'\,,$$

which will give the correct equations of motion providing

$$f'(x) = F'(x) \quad \Rightarrow \quad f(x) = F(x) + c_1\,, \qquad g''(\dot{x}) = \frac{1}{G(\dot{x})}\,. \tag{2.5}$$

The integration constant c_1 simply adds a trivial constant term to the Lagrangian, so we set it to zero.

[3]We are now tacitly introducing a notation for partial derivatives which we will make extensive use of throughout this section. A prime, unless otherwise stated indicates the partial derivative with respective to natural variable of the function, whereas a dot refers to a total time derivative. To wit:

$$F'(x) = \frac{\partial F}{\partial x}\,, \qquad \dot{F}(x) = \dot{x}F'(x)\,, \qquad G'(\dot{x}) = \frac{\partial G}{\partial \dot{x}}\,.$$

This is not unfamiliar notation, we simply needed to clarify.

Now consider a multiplicative ansatz,

$$\tilde{L} = \tilde{f}(x)\tilde{g}(\dot{x}).$$

This gives

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{x}}\right) = \frac{\partial \tilde{L}}{\partial x} \quad \Rightarrow \quad \tilde{f}'\tilde{g}'\dot{x} + \tilde{f}\tilde{g}''\ddot{x} = g\tilde{f}' \quad \Rightarrow \quad \ddot{x} = \frac{\tilde{f}'}{\tilde{f}}\frac{\tilde{g} - \tilde{g}'\dot{x}}{\tilde{g}''}.$$

In this case we see that we obtain the correct equations of motion if

$$\tilde{f}'(x) = \tilde{f}(x)F(x) \quad \Rightarrow \quad \tilde{f}(x) = Ae^{F(x)}, \qquad \tilde{g} - \tilde{g}'\dot{x} = G(\dot{x})\tilde{g}''. \tag{2.6}$$

Again, the integration constant doesn't affect the overall story, so we set $A = 1$. In order to actually implement this theory we would need to explicitly solve the ordinary differential equations in Equation 2.5 and Equation 2.6. Providing we were able to, this would give the Lutzky invariant

$$\Phi_L = \left(\frac{\partial^2 \tilde{L}}{\partial \dot{x}^2}\right)\left(\frac{\partial^2 L}{\partial \dot{x}^2}\right)^{-1} = \frac{\tilde{g}''}{g''}e^{F(x)} = \tilde{g}''e^{F(x)}G(\dot{x}) = (\tilde{g} - \tilde{g}'\dot{x})e^{F(x)}. \tag{2.7}$$

Again, we can rigorously demonstrate that this statement is indeed an invariant:

$$\begin{aligned}\frac{d\Phi_L}{dt} &= Ae^{F(x)}\left[F'(x)\dot{x}(\tilde{g} - \tilde{g}'\dot{x}) + \ddot{x}(\tilde{g}' - \tilde{g}''\dot{x} - \tilde{g}')\right] \\ &= A(\tilde{g} - \tilde{g}'\dot{x})\dot{x}e^{F(x)}\left[\frac{G(\dot{x})F'(x)}{G} - \frac{\ddot{x}}{G}\right] = A\frac{(\tilde{g} - \tilde{g}'\dot{x})}{G}\dot{x}e^{F(x)}\left[G(\dot{x})F'(x) - \ddot{x}\right] = 0.\end{aligned}$$

Evidently, all systems that are of the form $\ddot{x} = F'(x)G(\dot{x})$ will have an invariant of the form of Equation 2.7. This is powerful result, because without knowing the trajectory of the system, we were able to generate a conservation law.

However, the underlying reason that the object Φ_L in Equation 2.7 is invariant appears to be that it is related to the Hamiltonian associated with $\tilde{L}$, which has no explicit time dependence. In Section 1.2.2 we demonstrated that a conserved Hamiltonian is a Noether conserved quantity associated with time translation invariance. In this case the canonical momentum is given by

$$p = \frac{\partial \tilde{L}}{\partial \dot{x}} = \tilde{f}\tilde{g}',$$

which allows us to construct the Hamiltonian,

$$\tilde{H} = p\dot{x} - \tilde{L} = \tilde{f}\tilde{g}'\dot{x} - \tilde{f}\tilde{g} = \tilde{f}(\tilde{g}\dot{x} - \tilde{g}) = Ae^{F(x)}[\tilde{g}'(\dot{x})\dot{x} - \tilde{g}(\dot{x})]\,,$$

just as in Equation 2.7. This in turn implies that the class of invariants we have just developed are in fact all equivalent to Noether invariants.[4]

The ansatz technique makes quick work of generating Lutzky invariants in this and many other contexts. However, it doesn't reveal anything about the type of the transformation taking one Lagrangian to another. The presence of an invariant suggests that there is a relationship between them, but the relationship may be extremely cumbersome to describe.

Example: Simple Harmonic Oscillator

Let us consider a concrete example of a system where the ansatz technique gives us traction: the simple harmonic oscillator. As always, this system has the equation of motion $\ddot{x} = -\omega_0^2 x$. We note that it is clearly of the form we have an ansatz solution for, with $F'(x) = -\omega_0^2 x$ and $G(\dot{x}) = 1$. We will first generate a Lagrangian from an additive ansatz, as above. Applying Equation 2.5, we obtain

$$g'' = 1 \quad \Rightarrow \quad g = \frac{1}{2}\dot{x}^2 + c_1\dot{x} + c_2, \qquad f' = -\omega_0^2 x \quad \Rightarrow \quad f = -\frac{1}{2}\omega_0^2 x^2 + c_3\,.$$

Evidently our first Lagrangian is simply

$$L = \frac{1}{2}\dot{x}^2 + c_1\dot{x} + c_2 - \frac{1}{2}\omega_0^2 x^2 + c_3 = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega_0^2 x^2\,,$$

where we have set the constants of integration equal to zero, because they change the Lagrangian only by a total time derivative. This is the familiar Lagrangian that comes from the $L = T - U$ prescription. We now want to generate a second Lagrangian using multiplicative separation. Using Equation 2.6, we obtain in this case

$$\frac{\tilde{f}'}{\tilde{f}} = -x \quad \Rightarrow \quad \frac{\partial \tilde{f}}{\partial x} = -x\tilde{f} \quad \Rightarrow \quad \tilde{f}(x) = c_1 e^{-x^2/2}\,.$$

[4]In fact it is one of the most common things to find that the Lutzky Invariant is the Hamiltonian of one of the pair of Lagrangians.

Finding $\tilde{g}$ is somewhat more complicated. We start by changing variables $z = \dot{x}$:

$$\tilde{g} - \tilde{g}'z - \tilde{g}'' = 0 \quad \Rightarrow \quad \tilde{g}' - \tilde{g}' - \tilde{g}''z - \tilde{g}''' = 0 \quad \Rightarrow \quad -\tilde{g}''z = \tilde{g}''' \,.$$

While ugly, this does have a solution:

$$\tilde{g}(\dot{x}) = -c_2 e^{\dot{x}^2/2} + \dot{x}\left(c_1 - \sqrt{\frac{\pi}{2}} c_2 \mathrm{Erf}\left[\frac{\dot{x}}{2}\right]\right) \,.$$

We can now assemble these results into our second Lagrangian,

$$\tilde{L} = e^{-x^2/2}\left(-C_2 e^{\dot{x}^2/2} + \dot{x}\left(C_1 - \sqrt{\frac{\pi}{2}} C_2 \mathrm{Erf}\left[\frac{\dot{x}}{2}\right]\right)\right) \,.$$

While this Lagrangian may have an unintuitive form, a quick check in *Mathematica* shows that this does in fact yield the equations of motion. Finally, we are able to assemble the Lutzky Invariant:

$$\begin{aligned}\Phi_L &= \frac{\partial^2}{\partial \dot{x}^2}\left(e^{-x^2/2}\left(-C_2 e^{\dot{x}^2/2} + \dot{x}\left(C_1 - \sqrt{\frac{\pi}{2}} C_2 \mathrm{Erf}\left[\frac{\dot{x}}{2}\right]\right)\right)\right)\left[\frac{\partial^2}{\partial \dot{x}^2}\left(\dot{x}^2 - x^2\right)\right]^{-1} \\ &= \frac{C_1}{2} e^{-(x^2+\dot{x}^2)/2} \,.\end{aligned}$$

Just as we noted earlier, we have picked off an invariant related to the Hamiltonian, which in this case is the energy.

2.3.1 Extended Example: Lotka-Volterra

We will conclude this chapter by applying the techniques we have developed to the celebrated Lotka-Volterra model of Predator-Prey deterministic non-semelparous population dynamics. This is an interesting system for us because it consists of a pair of first-order linked ordinary differential equations, and allowing us to demonstrate how these techniques work in both nontraditional situations and situations when a greater number of dynamical variable are in play.

Because of the unfamiliarity of this system to many physicists, we will begin by reviewing its motivation. The most elementary version of this model describes the relationship between a carnivorous predator population (say, for example, foxes) and an herbivorous prey population (for example, rabbits). The basic motivation for the model is given by five assumptions:

- In absence of predators, the prey population will increase exponentially.
- In absence of prey, the predator population will decrease exponentially.
- The prey population decreases at a rate proportionate to the size of the population of predators.
- The predator population increases at a rate proportionate to the size of the population of prey.
- The system is closed, so that the species do not evolve or change the way in which they interact.

These assumptions lead to the following pair of ordinary differential equations, in which the populations of prey and predators are denoted X and Y respectively,

$$\dot{X} = X(a + bY)\,, \qquad \dot{Y} = Y(A + BX)\,, \tag{2.8}$$

where B and b represent the interaction parameters between the species, while a and A represent the natural re-population coefficients [25].[5,6] Because this system

[5]This model is part of a large collection of models known as the Kolmogorov models, which are given by

$$\dot{X} = XF(X,Y)\,, \qquad \dot{Y} = YG(X,Y)\,.$$

[6]In order to correctly model our assumptions we declare that a and B are greater than zero while A and b are less than zero.

is inherently deterministic, the dynamics of the system are fixed by the starting conditions and parameters of the system. The types of behavior that the system can take on are

1. Stable oscillatory paths, in which the populations fluctuate such that their phase space representation develops an orbit around a central point at $(X, Y) = (\frac{A}{B}, \frac{a}{b})$.
2. Extinction or unbounded growth, in which one of the populations reaches zero which causes the other to either become extinct, or grow indefinitely.

The dynamics of each of these cases can be seen in Figure 2.1.

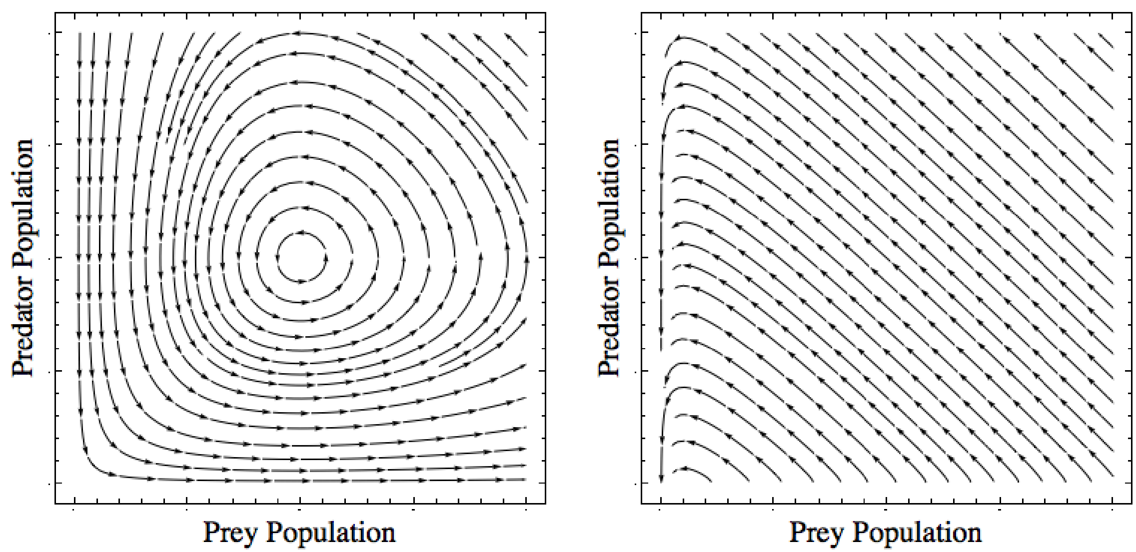

Figure 2.1: Flow density plots of the two types of dynamics available to this system. The vectors indicate the direction of the change in population at each point. The top graph shows regular oscillatory stability, while the graph on the bottom shows extinction.

If we make use of coordinate transformations we will be able to employ the tools of Nonequivalent Lagrangian Mechanics in this system. To do so we follow the suggestion of M.C. Nucci [23] and take a change of variables for Equation 2.8 such that

$$X = e^x \,, \qquad\qquad X = e^y \,.$$

This causes the equations of population to become

$$\dot{x} = a + be^y \,, \qquad\qquad \dot{y} = A + Be^x \,. \tag{2.9}$$

We now differentiate these to move our system into second-order form:

$$\ddot{x} = b\dot{y}e^y , \qquad \ddot{y} = B\dot{x}e^x ,$$

and then use Equation 2.9 to separate them:

$$\ddot{x} = b(A + Be^x)e^y = (A + Be^x)(\dot{x} - a) ,$$
$$\ddot{y} = B(a + be^y)e^x = (a + be^y)(\dot{y} - A) .$$

We have now totally separated the equations of population for this system, which are, coincidentally, of the precise form discussed in Section 2.3. Because the equations of population are now separated we will generate a total Lagrangian of the form $L = L_x(x, \dot{x}, t) + L_y(y, \dot{y}, t)$. Let's start with the x-dependent term. Using the methods of the previous section we see that an additive ansatz will give us the Lagrangian

$$L_{x1} = [Be^x + Ax] + [(\dot{x} - a)\ln(\dot{x} - a) - \dot{x}] ,$$

while a multiplicative ansatz will give us:

$$L_{x2} = \dot{x}e^{Ax+Be^x} \int_1^{\dot{x}} \frac{1}{s^2} e^{-s-a\ln(s-a)} ds .$$

These are naturally mirrored in y:

$$L_{y1} = [be^y + ay] + [(\dot{y} - A)\ln(\dot{y} - A) - \dot{y}] ,$$
$$L_{y2} = \dot{y}e^{ay+be^y} \int_1^{\dot{y}} \frac{1}{s^2} e^{-s-A\ln(s-A)} ds .$$

From these two pairs of partial Lagrangians, we can form four complete Lagrangians, which in principle yields four Jacobi Last Multipliers, and finally six Lutzky invariants. That being said, there is a lot of degeneracy among the invariants, so in the end we only have one interesting conserved quantity. In one case, the Lutzky Invariant is given as:

$$
\begin{aligned}
\Phi_L &= \det\begin{pmatrix} \frac{\partial^2 L_{x2+y1}}{\partial \dot{x}^2} & \frac{\partial^2 L_{x2+y1}}{\partial \dot{x}\partial \dot{y}} \\ \frac{\partial^2 L_{x2+y1}}{\partial \dot{y}\partial \dot{x}} & \frac{\partial^2 L_{x2+y1}}{\partial \dot{x}^2} \end{pmatrix} \left[\det\begin{pmatrix} \frac{\partial^2 L_{x1+y2}}{\partial \dot{x}^2} & \frac{\partial^2 L_{x1+y2}}{\partial \dot{x}\partial \dot{y}} \\ \frac{\partial^2 L_{x1+y2}}{\partial \dot{y}\partial \dot{x}} & \frac{\partial^2 L_{x1+y2}}{\partial \dot{x}^2} \end{pmatrix}\right]^{-1} \\
&= \det\begin{pmatrix} -(\dot{x}-a)^{-1-a}e^{Be^x+Ax-\dot{x}} & 0 \\ 0 & \frac{1}{\dot{y}-A} \end{pmatrix} \left[\det\begin{pmatrix} \frac{1}{\dot{x}-a} & 0 \\ 0 & (\dot{y}-A)^{-1-A}e^{be^y+ay-\dot{y}} \end{pmatrix}\right] \\
&= (\dot{x}-a)^{-a}(\dot{y}-A)^{A}e^{Be^x-be^y+Ax-ay-\dot{x}+\dot{y}} \,.
\end{aligned}
$$

We can simplify this expression by considering the equations of population, obtaining

$$
\begin{aligned}
\Phi_L &= (a+be^y-a)^{-a}(A+Be^x-A)^{A}e^{Be^x-be^y+Ax-ay-(a+be^y)+(A+Be^x)} \\
&= b^{-a}e^{-ay}B^{A}e^{Ax}e^{2Be^x-2be^y+Ax-ay-a+A} = b^{-a}B^{A}e^{A-a}e^{2(Ax+Be^x-ay-be^y)} \,.
\end{aligned}
$$

We can now transform the whole thing back into the original coordinates. The inverse transformation is given by

$$
x = \ln X, \qquad\qquad y = \ln Y \,.
$$

Thus our invariant in the original coordinates is:

$$
\Phi_L = b^{-a}B^{A}e^{A-a}e^{2(A\ln X+BX-a\ln Y-bY)} \,.
$$

This we can show to be invariant:

$$
\begin{aligned}
\frac{d\Phi_L}{dt} &= b^{-a}B^{A}e^{A-a}\frac{d}{dt}e^{2(A\ln X+BX-a\ln Y-bY)} \\
&= 2b^{-a}B^{A}e^{A-a}(\dot{X}(B+A\frac{1}{X})-\dot{Y}(b+a\frac{1}{Y}))e^{2(A\ln X+BX-a\ln Y-bY)} \\
&= 2b^{-a}B^{A}e^{A-a}(\frac{\dot{X}}{X}(A+BX)-\frac{\dot{Y}}{Y}(a+bY))e^{2(A\ln X+BX-a\ln Y-bY)} \\
&= \frac{2}{XY}(YX(a+bY)(A+BX)-XY(A+BX)(a+bY))\Phi_L = 0 \,.
\end{aligned}
$$

There is a well-known invariant for this system, which is usually presented without derivation, as something that can be simply guessed by study of the system [30]. It is given by

$$
h = BX + A\ln X - a\ln Y - bY \,.
$$

Thus Φ_L is simply the exponentiation of this, which is completely in line with our usual mode of results. By applying our machinery we have offered a concrete derivation.

3

Unexpected Noetherian Invariants

We will conclude the work of this thesis with an in depth investigation of a system which in not traditionally thought of as having Noetherian invariants: the damped driven harmonic oscillator. To do so, we will make use of a technique developed by Curie and Saletan[1] in [6], in which an invariant is guessed from the form of an initial Lagrangian and knowledge of the equations of motion. This result is then confirmed in the context of the Lutzky invariant machinery by constructing a second Lagrangian, and then in the context of Noetherian analysis using traditional methods.

We will motivate the technique of Curie and Saletan by using it to reproduce a number of results pertaining to the damped harmonic oscillator that are known from the literature ([1], [26], [28]). Once the tools are in hand we will approach the driven damped harmonic oscillator, which the literature does not appear to have anything to say about.

3.1 Damped Harmonic Oscillator

We begin by considering a traditional system in undergraduate mechanics, the damped harmonic oscillator. The system is dissipative, so neither energy nor momentum are conserved. Its equation of motion is:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0 \,, \tag{3.1}$$

[1]We stumbled upon this procedure prior to reading the work of Curie and Saletan.

where ω_0 is the natural frequency and γ is the dampening parameter. We propose an ansatz,

$$L = h(t)\Big(f(x) + g(\dot{x})\Big).$$

Applying the Euler-Lagrange equations gives:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \frac{d}{dt}\left(hg'\right) - hf' = h'g' + hg''\ddot{x} - hf' = 0 \quad \Rightarrow \quad \ddot{x} = \frac{f'}{g''} - \frac{h'}{h}\frac{g'}{g''}\,.$$

Forcing these terms to conform to Equation 3.1 gives the constraints

$$f' = \omega_0^2 x, \qquad g'' = 1, \qquad h' = \gamma h, \qquad \frac{g'}{g''} = -\dot{x}\,,$$

which have solutions

$$f = \frac{1}{2}\omega_0^2 x^2\,, \qquad g = \frac{1}{2}\dot{x}^2\,, \qquad h = e^{\gamma t}\,.$$

This yields the Lagrangian

$$L = e^{\gamma t}(\dot{x}^2 - \omega_0^2 x^2)\,. \tag{3.2}$$

Here we are, as usual, setting irrelevant constants of integration to zero or one to put our Lagrangian in the simplest form possible. This nicely agrees with the Lagrangian found in [26]. We can also verify that Equation 3.2 yields the correct equation of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(2\dot{x}e^{\gamma t}\right) + 2\omega_0^2 x e^{\gamma t} = 0 \qquad \Rightarrow \qquad \ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0\,.$$

We now need to solve the equations of motion, following the familiar procedure of taking an ansatz: $x = Ae^{\rho t}$, where ρ is a undetermined constant. By plugging this into our equations of motion, we obtain,

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \quad \Rightarrow \quad A\rho^2 e^{\rho t} + \gamma A\rho e^{\rho t} + \omega_0^2 A e^{\rho t} = 0 \quad \Rightarrow \quad \rho^2 + \gamma\rho + \omega_0^2 = 0\,.$$

There are two solutions to this algebraic equation:

$$\rho_{\pm} = \frac{1}{2}\left(-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}\right)\,. \tag{3.3}$$

In order to completely span the solution space of the differential equation, we write our full solution as a linear combination of the solutions developed by the algebraic equation:

$$x = Ae^{\rho_+ t} + Be^{\rho_- t}\,, \qquad\qquad \dot{x} = A\rho_+ e^{\rho_+ t} + B\rho_- e^{\rho_- t}\,.$$

Again, our project here is to find constants of the motion for this system, so we need to isolate the integration constants A and B in terms of x, $\dot{x}$, and t. We can rewrite our system using matrix multiplication as

$$\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} e^{\rho_+ t} & e^{\rho_- t} \\ \rho_+ e^{\rho_+ t} & \rho_- e^{\rho_- t} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

This allows us to take the matrix inverse and solve for A and B:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{(\rho_- - \rho_+)e^{(\rho_- + \rho_+)t}} \begin{pmatrix} \rho_- e^{\rho_- t} & -e^{\rho_- t} \\ -\rho_+ e^{\rho_+ t} & e^{\rho_+ t} \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix},$$

which gives

$$A = \frac{\rho_- x - \dot{x}}{(\rho_- - \rho_+)e^{\rho_+ t}}\,, \qquad B = \frac{-\rho_+ x + \dot{x}}{(\rho_- - \rho_+)e^{\rho_- t}}\,.$$

These are "constants of the motion" set by the initial conditions, so any combination of them is also constant. A particularly useful combination is their product:

$$\begin{aligned} AB &= \left(\frac{\rho_- x - \dot{x}}{(\rho_- - \rho_+)e^{\rho_+ t}}\right)\left(\frac{-\rho_+ x + \dot{x}}{(\rho_- - \rho_+)e^{\rho_- t}}\right) \\ &= -\left(\frac{\rho_+\rho_- x^2 - (\rho_+ + \rho_-)x\dot{x} + \dot{x}^2}{(\rho_- - \rho_+)^2}\right) e^{-(\rho_+ + \rho_-)t}\,. \end{aligned} \tag{3.4}$$

There are a number of combinations of ρ_+ and ρ_- that would be simpler if we rewrote them in terms of the original parameters of the problem:

$$\rho_- + \rho_+ = -\gamma\,, \qquad \rho_- - \rho_+ = \sqrt{\gamma^2 - 4\omega_0^2}\,, \qquad \rho_-\rho_+ = \omega_0^2\,.$$

Thus Equation 3.4 becomes:

$$AB = -\frac{\omega_0^2 x^2 + \gamma x\dot{x} + \dot{x}^2}{(\sqrt{\gamma^2 - 4\omega_0^2})^2} e^{\gamma t} = -(\omega_0^2 x^2 + \gamma x\dot{x} + \dot{x}^2)\frac{e^{\gamma t}}{\gamma^2 - 4\omega_0^2}\,. \tag{3.5}$$

This result agrees with the invariant found in [1]. We want to show that this is a Lutzky invariant, so we must now construct a second Lagrangian. We start by assuming,

$$\Phi_L = AB = \frac{\tilde{M}}{M} \qquad \Rightarrow \qquad M\Phi_L = \tilde{M}\,.$$

Here we use our first Lagrangian to generate $\tilde{M}$:

$$\tilde{M} = \frac{\partial^2 \tilde{L}}{\partial \dot{x}^2} = 2e^{\gamma t}\,,$$

and this implies that

$$-(\omega_0^2 x^2 + \gamma x\dot{x} + \dot{x}^2)\frac{e^{\gamma t}}{\gamma^2 - 4\omega_0^2}M = 2e^{\gamma t} \qquad \Rightarrow \qquad M = \frac{\partial^2 L}{\partial \dot{x}^2} = \frac{-2(\gamma^2 - 4\omega_0^2)}{\omega_0^2 x^2 + \gamma x\dot{x} + \dot{x}^2}\,.$$

By direct integration we can compute the Lagrangian L that gives the desired Jacobi Last Multiplier, M:

$$\frac{\partial^2 L}{\partial \dot{x}^2} = \frac{-(\gamma^2 - 4\omega_0^2)}{\omega_0^2 x^2 + \gamma x\dot{x} + \dot{x}^2} \qquad \Rightarrow \qquad \frac{\partial L}{\partial \dot{x}} = \frac{2}{x}\tanh^{-1}\left[\frac{\gamma x + 2\dot{x}}{\sqrt{4\omega^2 - \gamma^2}x}\right]\,,$$

$$L = \sqrt{4\omega_0^2 - \gamma^2}\left(\frac{2\dot{x}+\gamma x}{x}\tanh^{-1}\left[\frac{\gamma x + 2\dot{x}}{\sqrt{4\omega^2-\gamma^2}x}\right] - \frac{1}{2}\sqrt{4\omega_0^2-\gamma^2}\ln\left[\omega^2 x^2 + \gamma x\dot{x} + \dot{x}^2\right]\right)\,. \qquad (3.6)$$

This Lagrangian agrees with the results of [28]. Again, we are fixing the constants of integration to put this in the simplest form possible.

However, the assumption that the constants of integration are simply detritus can accidentally yield false positives and as a resut this procedure can produce a second Lagrangian that does not give the correct equations of motion.[2] We generally need to check that the new Lagrangian is valid. For Equation 3.6, we have

[2]Nucci [23] contends that this error can be avoided by the integration scheme

$$L = \int M d\dot{x} d\dot{x} + \frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} + l_3$$

Where F is an arbitrary function of x and t (but not $\dot{x}$), and l_3 is related to the definition of the Jacobi Last Multiplier.

$$\begin{aligned}
\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] &= \frac{d}{dt}\left[\frac{2\sqrt{4\omega^2-\gamma^2}}{x}\tanh^{-1}\left[\frac{\gamma x + 2\dot{x}}{\sqrt{4\omega^2-\gamma^2}x}\right]\right] \\
&= \frac{\gamma^2-4\omega^2}{x} - \frac{2\sqrt{4\omega^2-\gamma^2}\dot{x}}{x^2}\tanh^{-1}\left[\frac{\gamma x + 2\dot{x}}{\sqrt{4\omega^2-\gamma^2}x}\right] \\
&- \frac{(\gamma^2-4\omega^2)(\omega^2 x + \gamma\dot{x} + \ddot{x})}{\omega^2 x^2 + \gamma x\dot{x} + \dot{x}^2}, \\
\frac{\partial L}{\partial x} &= \frac{\gamma^2-4\omega^2}{x} - \frac{2\sqrt{4\omega^2-\gamma^2}\dot{x}}{x^2}\tanh^{-1}\left[\frac{\gamma x + 2\dot{x}}{\sqrt{4\omega^2-\gamma^2}x}\right], \\
0 &= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] - \frac{\partial L}{\partial x} = -\frac{(\gamma^2-4\omega^2)(\omega^2 x + \gamma\dot{x} + \ddot{x})}{\omega^2 x^2 + \gamma x\dot{x} + \dot{x}^2}, \\
\Rightarrow \qquad & 0 = \omega^2 x + \gamma\dot{x} + \ddot{x}\,.
\end{aligned}$$

Through simple observation we can see that $\tilde{L}$ is independent of time, which means that it has a corresponding conserved Hamiltonian, $\tilde{H}$. In fact, the invariant found in Equation 3.5 is related to $\tilde{H}$. This agrees with the major thrust of [1], which demonstrates through direct application of Noether's theorem that the damped harmonic oscillator has a Noether invariant.

3.2 Damped Driven Harmonic Oscillator

The natural follow up to the damped harmonic oscillator is the driven damped harmonic oscillator. We will consider a sinusoidal driving term at the same frequency as the natural frequency of the system,[3] which gives us the equation of motion

$$\ddot{x} = -\omega_0^2 x - \gamma\dot{x} + f_0\cos(\omega_0 t)\,.$$

We begin by proposing a time-dependent Lagrangian, of a similar form as our initial Lagrangian in the previous section,

$$L = e^{\gamma t}\left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega_0^2 x^2 + f_0 x\cos(\omega_0 t)\right).$$

[3] The same story holds for a driving frequency that is not the same as the natural frequency, but we will present this case here and leave it to the reader to verify that the more general case in Mathematica.

This gives the equation of motion

$$\begin{aligned}\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] - \frac{\partial L}{\partial x} &= e^{\gamma t}\ddot{x} + \gamma e^{\gamma t}\dot{x} - e^{\gamma t}(-\omega_0^2 x + f_0\cos(t)) \\ &= \ddot{x} + \omega_0^2 x + \gamma\dot{x} - f_0\cos(\omega_0 t) = 0\,,\end{aligned}$$

just as we expect it to be. We now need to solve this system, in this case using the method of undetermined coefficients, where we propose an ansatz based on the form of the non-homogenous term in the differential equation. Here the non-homogenous term is $f_0\cos(\omega_0 t)$, so we propose $x = A\cos(\omega_0 t) + B\sin(\omega_0 t)$. This leads to the complete solution:

$$x(t) = Ae^{\rho_+ t} + Be^{\rho_- t} + \frac{f_0}{\gamma\omega_0}\sin(\omega_0 t)\,, \qquad x'(t) = A\rho_+ e^{\rho_+ t} + B\rho_- e^{\rho_- t} + \frac{f_0}{\gamma}\cos(\omega_0 t)\,,$$

where, $\rho_\pm$ are defined as in Equation 3.3. We can rewrite these expression as a matrix equation

$$\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} e^{\rho_+ t} & e^{\rho_- t} \\ \rho_+ e^{\rho_+ t} & \rho_- e^{\rho_- t} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} + \frac{f_0}{\gamma\omega_0}\begin{pmatrix} \sin(\omega_0 t) \\ \omega_0\cos(\omega_0 t) \end{pmatrix}\,,$$

and therefore we have

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{e^{-t(\rho_+ + \rho_-)}}{(\rho_+ - \rho_-)}\begin{pmatrix} \rho_- e^{\rho_- t} & -e^{\rho_- t} \\ -\rho_+ e^{\rho_+ t} & e^{\rho_+ t} \end{pmatrix}\left[\begin{pmatrix} x \\ \dot{x} \end{pmatrix} - \frac{f_0}{\gamma\omega_0}\begin{pmatrix} \sin(\omega_0 t) \\ \omega_0\cos(\omega_0 t) \end{pmatrix}\right]\,.$$

Next we construct $\Phi = AB$ as our proposed Lutzky invariant, just as before:

$$\begin{aligned}\Phi = \frac{e^{\gamma t}}{\gamma\omega_0(\gamma - 4\omega_0^2)}\Big(&f_0\omega_0\cos(\omega_0 t)(\gamma x + 2\dot{x}) + f_0\sin(\omega_0 t)(2\omega_0^2 x + \gamma\dot{x}) \\ &- \gamma\omega_0(\omega_0 x^2 + \gamma x\dot{x} + \dot{x}^2) - \frac{f_0^2}{2\gamma}(2\omega_0 + \gamma\sin(2t\omega_0))\Big)\,.\end{aligned} \tag{3.7}$$

We then build our second Lagrangian by making use of $\Phi = AB = M/\tilde{M}$, with

$$\tilde{M} = \frac{\partial^2\tilde{L}}{\partial\dot{x}^2} = e^{\gamma t}\,.$$

This allows us to compute $\tilde{M}$, by manipulating

$$\tilde{M} = \frac{\partial^2\tilde{L}}{\partial\dot{x}^2} = \frac{M}{AB} = \frac{1}{AB}\frac{\partial^2 L}{\partial\dot{x}^2}\,,$$

so that

$$\frac{\partial^2 L}{\partial \dot{x}^2} = \frac{\gamma\omega_0(\gamma-4\omega_0^2)}{\left(f_0\omega_0\cos(\omega_0 t)(\gamma x+2\dot{x})+f_0\sin(\omega_0 t)(2\omega_0^2 x+\gamma\dot{x})-\gamma\omega_0(\omega_0 x^2+\gamma x\dot{x}+\dot{x}^2)-\frac{f_0^2}{2\gamma}(2\omega_0+\gamma\sin(2t\omega_0))\right)}\,.$$

While this is menacing, it turns out that it is in fact analytically tractable. Repeated integration gives the Lagrangian

$$\begin{aligned} L = \frac{\sqrt{\gamma^2-4\omega_0^2}}{(\gamma\omega_0 x - f_0\sin(\omega_0 t))}\Bigg(& f_0(2\omega_0\cos(\omega_0 t)+\gamma\sin(\omega_0 t))\tanh^{-1}\left[\frac{2f_0\omega_0\cos(\omega_0 t)+f_0\gamma\sin(t\omega_0)-\gamma\omega_0(\gamma x+2\dot{x})}{\sqrt{\gamma^2-4\omega_0^2}(-f_0\sin(\omega_0 t)+\gamma\omega_0 x)}\right] \\ & +\gamma\omega_0(\gamma x+2\dot{x})\tanh^{-1}\left[\frac{2f_0\omega_0\cos(\omega_0 t)+f_0\gamma\sin(t\omega_0)-\gamma\omega_0(\gamma x+2\dot{x})}{\sqrt{\gamma^2-4\omega_0^2}(f_0\sin(\omega_0 t)+\gamma\omega_0 x)}\right]\Bigg) \\ & + (\gamma^2-4\omega_0^2)\ln\big[f_0^2(2\omega_0+\gamma\sin(2t\omega_0)) - 2\gamma(f_0\omega_0\cos(\omega_0 t)(\gamma x+2\dot{x}) \\ & \quad +f_0\sin(\omega_0 t)(2\omega_0^2 x+\gamma\dot{x}) - \gamma\omega_0(\omega_0^2 x^2+\gamma x\dot{x}+\dot{x}^2))\big]\ . \end{aligned} \tag{3.8}$$

By checking *Mathematica*, we can see that this Lagrangian yields the right equation of motion. This demonstrates that the quantity constructed in Equation 3.7 is a Lutzky Invariant. It is pleasantly reassuring that we recover the invariant that we derived for the DHO when we set f_0 to zero. This procedure also works for the a time-dependent driving force of $f_0\cos[\omega(t)t]$, however it is a quite cumbersome thing to write down, so again we, refer the reader to *Mathematica*.

The interesting thing about these Lagrangians is that each of them depend explicitly on time, so neither has an associated conserved Hamiltonian. While it is tempting to assume that this implies the quantity that we have developed is not related to any Noether invariant, as we will see in the next section, this is not the case.

3.2.1 Noether Analysis

Now consider this system from the perspective of Noether analysis, for which we can roughly follow the techniques presented in [1].[4] Again we will assume the driving frequency and the natural frequency are the same. We begin by writing down the form of the Noether invariant for this system:

$$\begin{aligned} \Phi_N &= (\tau\dot{x} - \eta)\frac{\partial L}{\partial \dot{x}} - \tau L + J \\ &= (\tau\dot{x} - \eta)\left[\dot{x}e^{\gamma t}\right] - \tau e^{\gamma t}\left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega_0^2 x^2 + f_0 x\cos\omega_0 t\right) + J\,. \end{aligned} \tag{3.9}$$

where J, η, and τ are unknown functions of x, $\dot{x}$, and t. By assumption this is an invariant, so the total time derivative must be equal to zero. This is gives us

$$\begin{aligned} \frac{d\Phi_N}{dt} &= (\dot{\tau}\dot{x} + \tau\ddot{x} - \dot{\eta})\dot{x}e^{\gamma t} + (\tau\dot{x} - \eta)\ddot{x}e^{\gamma t} + (\tau\dot{x} - \eta)\dot{x}\gamma e^{\gamma t} \\ &- \dot{\tau}e^{\gamma t}\left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega_0^2 x^2 + f_0 x\cos\omega t\right) - \tau\gamma e^{\gamma t}\left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega_0^2 x^2 + f_0 x\cos\omega_0 t\right) \\ &- \tau e^{\gamma t}\left(\dot{x}\ddot{x} - \omega_0^2 x\dot{x} + f_0\dot{x}\cos\omega_0 t - f_0\omega_0 x\sin\omega_0 t\right) + \frac{dJ}{dt} \\ &= -f_0\cos(\omega_0 t)\eta + \omega_0^2 x\eta - f_0\gamma\cos(\omega_0 t)x\tau + f_0\omega_0\sin(\omega_0 t)x\tau + \frac{1}{2}\gamma\omega_0^2 x^2\tau \\ &- \frac{1}{2}\gamma\tau\dot{x}^2 - \dot{x}\frac{\partial\eta}{\partial t} - \dot{x}^2\frac{\partial\eta}{\partial x} + \left[-f_0\cos(\omega_0 t)x + \frac{1}{2}\omega_0^2 x^2\right]\frac{\partial\tau}{\partial x}\dot{x} + \frac{1}{2}\dot{x}^3\frac{\partial\tau}{\partial x} \\ &+ \left[-f_0\cos(\omega_0 t)x + \frac{1}{2}\omega_0^2 x^2\right]\frac{\partial\tau}{\partial t} + \frac{1}{2}\dot{x}^2\frac{\partial\tau}{\partial t} + \frac{\partial J}{\partial t}e^{-t\gamma} + \frac{\partial J}{\partial x}\dot{x}e^{-t\gamma}\,. \end{aligned}$$

[4]We will be performing this analysis on the original Lagrangian we wrote down for this system rather than the one we derived. We invite the reader to try their hand at this type of analysis for the Lagrangian found in Equation 3.8, however we suggest that such a project might require the time remaining until the heat death of the universe.

Following the technique of [1], we insist that the coefficients of each power of $\dot{x}$ must separately equal zero. This gives us four equations

$$\dot{x}^0 \left[\omega_0^2 x\eta + \frac{1}{2}\omega_0^2 x^2\left(\gamma\tau + \frac{\partial\tau}{\partial t}\right) - f_0\left(\eta + \gamma x\tau + x\frac{\partial\tau}{\partial t}\right)\cos(\omega_0 t) + f_0\omega_0\sin(\omega_0 t)x\tau + \frac{\partial J}{\partial t}e^{-t\gamma}\right] = 0\,, \tag{3.10}$$

$$\dot{x}^1 \left[-\frac{\partial\eta}{\partial t} + \left[-f_0\cos(\omega_0 t)x + \frac{1}{2}\omega_0^2 x^2\right]\frac{\partial\tau}{\partial x} + \frac{\partial J}{\partial x}e^{-t\gamma}\right] = 0\,, \tag{3.11}$$

$$\dot{x}^2\left[-\frac{1}{2}\gamma\tau - \frac{\partial\eta}{\partial x} + \frac{1}{2}\frac{\partial\tau}{\partial t}\right] = 0\,, \tag{3.12}$$

$$\frac{1}{2}\dot{x}^3\left[\frac{\partial\tau}{\partial x}\right] = 0\,. \tag{3.13}$$

From Equation 3.13, we see that τ must be solely dependent on time, which we will now denote $\tau = \beta(t)$. This allows us to derive η from Equation 3.12:

$$\frac{\partial\eta}{\partial x} = \frac{1}{2}\left(\dot{\beta} - \gamma\beta\right) \quad\Rightarrow\quad \eta = \frac{1}{2}\left(\dot{\beta} - \gamma\beta\right)x + \psi(t) = \frac{1}{2}\dot{\beta}x - \frac{1}{2}\gamma\beta x + \psi(t)\,,$$

where $\psi(t)$ is a constant of integration. This in turn allows us to compute the Noether current from Equation 3.11:

$$\frac{\partial J}{\partial x} = \frac{\partial\eta}{\partial t}e^{t\gamma} = \left(\frac{1}{2}\left(\ddot{\beta} - \gamma\dot{\beta}\right)x + \dot{\psi}\right)e^{t\gamma} \quad\Rightarrow\quad J = \left(\frac{1}{4}\left(\ddot{\beta} - \gamma\dot{\beta}\right)x^2 + \dot{\psi}x\right)e^{t\gamma} + \xi(t)\,,$$

where $\xi(t)$ is a second constant of integration. We then insert these quantities into Equation 3.10:

$$\begin{aligned}0 &= \omega_0^2\left(\frac{1}{2}\dot{\beta}x^2 - \frac{1}{2}\gamma\beta x^2 + x\psi(t)\right) + \frac{1}{2}\omega_0^2 x^2\left(\gamma\beta + \dot{\beta}\right)\\ &- f_0\left(\frac{3}{2}\dot{\beta}x + \frac{1}{2}\gamma\beta x + \psi(t)\right)\cos(\omega_0 t) + \left(\frac{1}{4}\left(\ddot{\beta} - \gamma\dot{\beta}\right)x^2 + \dot{\psi}x\right)\gamma\\ &+ \left(\frac{1}{4}\left(\dddot{\beta} - \gamma\ddot{\beta}\right)x^2 + \ddot{\psi}x + \dot{\xi}e^{-\gamma t}\right) + f_0\omega_0\sin(\omega_0 t)x\beta\,.\end{aligned}$$

Repeating the technique we just used to derive Equations 3.10−3.13, we break this into three equations with different dependencies on x, each of which must separately vanish:

$$x^0 \left[\dot{\xi} e^{-\gamma t} - f_0 \psi(t) \cos(\omega_0 t) \right] = 0 \,, \tag{3.14}$$

$$x^1 \left[\omega_0^2 \psi(t) + \dot{\psi}\gamma + \ddot{\psi} + f_0 \omega_0 \sin(\omega t)\beta - \frac{1}{2} f_0 \left(3\dot{\beta} + \gamma\beta \right) \cos(\omega_0 t) \right] = 0 \,, \tag{3.15}$$

$$\frac{1}{2} x^2 \left[\left(4\omega_0^2 - \gamma^2 \right) \dot{\beta} + \dddot{\beta} = 0 \right] = 0 \,. \tag{3.16}$$

The complete solution to Equation 3.16 is given by

$$\beta = \frac{A}{\mu} e^{t\mu} + \frac{B}{\mu} e^{-t\mu} + C \,,$$

where $\mu = \sqrt{\gamma^2 - 4\omega_0^2}$. However, in interest of simplicity we will only study the solution $\beta = C = 1$. This causes Equation 3.15 to become:

$$\omega_0^2 \psi(t) + \dot{\psi}\gamma + \ddot{\psi} + f_0 \omega_0 \sin(\omega_0 t) - \frac{1}{2} f_0 \gamma \cos(\omega_0 t) = 0 \,,$$

which is just the equation of motion for another damped driven harmonic oscillator. This has complete solution,

$$\psi(t) = C_1 e^{\frac{t}{2}(-\gamma - \mu)} + C_2 e^{\frac{t}{2}(-\gamma + \mu)} + \frac{f_0}{\gamma} \cos(\omega_0 t) + \frac{f_0}{2\omega_0} \sin(\omega_0 t) \,.$$

We will continue our interest in simplicity and specialize to the particular solution in which $C_1 = C_2 = 0$. Plugging this into Equation 3.14 gives us an expression for ξ

$$\dot{\xi} = f_0^2 e^{\gamma t} \cos(\omega_0 t) \left(\frac{1}{\gamma} \cos(\omega_0 t) + \frac{1}{2\omega_0} \sin(\omega_0 t) \right) \,.$$

This also has a closed form solution, which is given by

$$\xi(t) = \frac{f_0^2 e^{\gamma t}}{2\gamma^2 \omega_0} \left(\omega_0 + \frac{\gamma}{2} \sin(2\omega_0 t) \right) + C_3 \,.$$

One last time we choose the simplest option: $C_3 = 0$. We now explicitly form our transformations:

$$\eta = -\frac{1}{2}\gamma x + \frac{f_0}{\gamma} \cos(\omega_0 t) + \frac{f_0}{2\omega_0} \sin(\omega_0 t) \,, \qquad \tau = 1 \,,$$

as well as the associated Noether current,

$$J = f_0 \left(\frac{1}{2} \cos(\omega_0 t) - \frac{\omega_0}{\gamma} \sin(\omega_0 t) \right) x e^{t\gamma} + \frac{f_0^2 e^{\gamma t}}{2\gamma^2 \omega_0} \left(\omega_0 + \frac{\gamma}{2} \sin(2\omega_0 t) \right) .$$

By plugging these expressions into Equation 3.9 we can obtain the Noether conserved quantity:

$$\begin{aligned} \Phi_N = \frac{e^{\gamma t}}{2\gamma\omega_0} \Big[& \gamma\omega_0 \left(\omega_0^2 x^2 + \dot{x}^2 + \gamma x \dot{x} \right) - (2\dot{x} + \gamma x) f_0 \cos(\omega_0 t) \\ & + \frac{f_0^2}{2\gamma} (2\omega_0 + \gamma \sin(2\omega_0 t)) - f_0 \left(2\omega_0^2 x + \gamma \dot{x} \right) \sin(\omega_0 t) \Big] . \end{aligned}$$

This recovers the Lutzky Invariant developed in Equation 3.7, up to a multiplicative factor. Specifically, we have

$$\Phi_N = -\frac{(\gamma^2 - 4\omega_0^2)}{2} \Phi_L .$$

We have thus demonstrated that the Lutzky invariant gives results that are equivalent to the results of Noether's theorem in the case of the damped driven harmonic oscillator. This is completely characteristic of our usual results and continues to speak to the power of Noether's theorem. It appears as though this is evocative of the general behavior of the Lutzky invariant, that it is generally a Noether invariant in disguise.

4

Conclusion

4.1 Overview

We set out to study a brand of classical mechanics that promised a new perspective on the construction and manipulation of invariants in dynamical systems. In the process, we considered a wide variety of cases, both familiar and unfamiliar, and in almost every one found that our Nonequivalent Lagrangian Mechanics made quick work of generating conserved quantities, but they were invariably related to Noether invariants. At the core of this study was a meditation on the interaction between NLM and Noether's Theorem. The two seem to be deeply and intrinsically linked, but due to the abstract nature of each of them, their exact relationship is still veiled.

We started by reviewing the classical mechanics background that was necessary to construct Noether's theorem in its simplest perturbative form. Using this construction we were able to derive a couple of traditional conserved quantities.

We then introduced the principal theorem behind NLM, as well as as the standard approach to generating nonequivalent Lagrangians via infinitesimal transformations. After demonstrating the holes in that procedure, we proceeded to develop our own ansatz-based approach to generating Lagrangians, which we used to explore the well known, but poorly motivated, conservation law of the Lotka-Volterra Predator-Prey population dynamics system.

Finally, we engaged in an extended case study of the damped driven harmonic oscillator. We demonstrated that this system has an associated Lutzky invariant, which we then rigorously demonstrated to be equivalent to a Noether Invariant.

After reviewing dozens of cases, and constructing endless Lutzky invariants, we see no evidence of conservation laws that are not related to Noetherian quantities. The lack of robust methodology for easily determining whether or not a invariant has an associated Noether symmetry is frustrating. Yet, despite this lack of understanding in the foundation, the theory that has been developed here yields a strong set of tools for rapidly finding conserved quantities.

4.2 Future work with NLM

Throughout the process of researching this project, we became rather taken by a wide variety of tangents to the main work. If time permitted they would provide excellent applications for Lutzky's theory of invariants.

It would be interesting to develop a version of this theory for discretized space. One of the current positions on the topic says that there cannot be a Noether Law for discrete spaces because the action cannot be conserved across transformation. [3] However this may present no problem for the Lutzky machinery. The most direct application of this new theory would be to develop a theory of conservation laws for cellular automata. [14]

In [15], J R Farias and N L Teixeira start to approach a field theoretic version of this brand of mechanics. However, they are plagued by the same issue that exists in most of the work on this topic: they create a mathematical formalism but do not use it do any meaningful work. There are a wide variety of conceivable applications for a complete version of their formalism, but we suggest that the most interesting one might to give a more direct derivation of the Kerr metric than is commonly available.[7]

It is also reasonably well known that one can write down a Lagrangian that yields the Schrödinger Equation. Using this knowledge it might be possible to generate an additional Lagrangian, and thereby engage the machinery of Lutzky. The original purpose of the NLM theory[1] was in fact to approach quantum theory from a novel direction, however their work was primarily based on a Hamiltonian version of the theory, which unfortunately did not generate any substantial results. With the machinery that has been developed for this theory over the past fifty years, we suggest that unusual results for quantum mechanics might be possible.

[1]Found in the work Currie and Saletan from 1966.

The ambiguity that is built into the Lagrangian representation of dynamical systems, allowing for multiple Lagrangians to be constructed for the same dynamical system, opens a floodgate for new insights into these systems. The limits of these applications are not clear, but in their boundless allure and ability to give new perspectives on systems, *we want to believe*.

A

Literature Review

One of the central difficulties in completing the research for this project has been the disunity of both name and content across the many sources. There are a wide variety of names for the topic, and an endless set of different but related approaches, and very little communication between the authors.

However, the main narrative of Nonequivalent Lagrangian Mechanics is reasonably simple, and its relationship to this thesis can be broken roughly into five pieces. These divisions and the relationships between them can be seen in Figure A.1. We note that throughout the literature authors reference so called Non-Noether invariants, however, as we have discussed, these invariants tend to be just esoteric Noether conservation laws in disguise.

Precursors:

There are two main papers that allowed the work of NLM to happen. Chronologically first, in 1941 Douglas[5] developed a set of criteria, known as the Helmholtz conditions, that described which dynamical systems could have a Lagrangian representation. This work discussed a relative ambiguity in the form of the Lagrangian that would eventually lead to our work.

In 1966 Currie and Saletan[1] published a paper [6] which described a method for manipulating this ambiguity, with the end goal of making new predictions about quantum mechanics. This paper was poorly received and was largely ignored for most of the following decade.

NLM:

There are a wide variety of authors that have discussed the premises of manipulating pairs of Lagrangians to generate invariants. In this thesis we have

[1] Who referred to their work as qEquivilance.

primarily focused on the work of M. Lutzky because we came across his work first, and so have taken it as cannon.

In *Lutzky 1978*[17] our author developed a notion of infinitesimal transformations and their relationship to dynamical systems, and applied it to the simple harmonic oscillator. It was not until *Lutzky 1979a*[18] that the notion of the Lutzky invariant was developed. In *Lutzky 1979b*[19] he presented a more concrete proof of the invariant. Finally in *Lutzky 1981*[20] he presented a case study describing a discrete symmetry leading to a particular conservation law for the free particle, and suggested that the general invariant he developed offers a perspective into a wider regime of invariants than Noether gives access to.

Alternative Perspectives:

There are a wide variety of additional authors who worked on this topic, whose work describes an equally complete cannon. Principle among these alternative perspectives are the works of Hojman[11][2], Sarlet[29], and Crampin[4]. In the period of 1980-1983, each of these authors developed their own theory of the relationship between Non-Noether Invariants and Lagrangians.

The most important of these works was *Crampin 1983*[4] in which Crampin took up the work of approaching the topic from a differential geometry perspective. He proved that most of the work on NLM had been trivial in light of the deeper framework. Following the publication of this article the interest in the topic pretty well died out. Authors such as Lutzky, Hojman, and others would still publish an occasional article on the matter, such as *Hojman 92*[12] or *Lutzky 95* [21], however the works would appear in more and more obscure journals.

Dissipative Systems:

While not completely essential to the main narrative of NLM, there have been a variety of approaches to studying the mechanics of dissipative systems using some of the theories of nonequivalence. In our work we consider some of the works of Subrata Ghosh and collaborators, who discuss, in *Ghosh 2004*[1] and *Ghosh 2007* [28], the nontraditional features of the damped harmonic oscillator. The approach in [1] primarily focuses on symmetry, while the latter is more centered on being able to generate endless collections of Lagrangians. On a similar tangent we also considered the works of M.C. Nucci, whom we discuss in the context of the Lotka-Volterra system. In her huge variety of papers on

[2] Who refers to the project as either Equivalent or s-Equivalent

the topic of non-traditional Lagrangian analysis, such as *Nucci 2012*[23], she generates enormous families of Lagrangians for a wide range of cases, including dissipative system[22], and other biological systems.

Modern Perspectives:

Despite the main period of interest in this theory being pretty well over, interest will occasionally flair up for a paper or two. The main modern perspective that we looked at in our work is in the writings of Zhang [13], which offers a pretty generic view on the types of invariants constructed by using infinitesimal transformations. Additionally, F.X. Mei has been quite productive in recent years in his works about new types of symmetries that generate what he claims are non-Noether invariants, such as in *Mei 2013*[8].

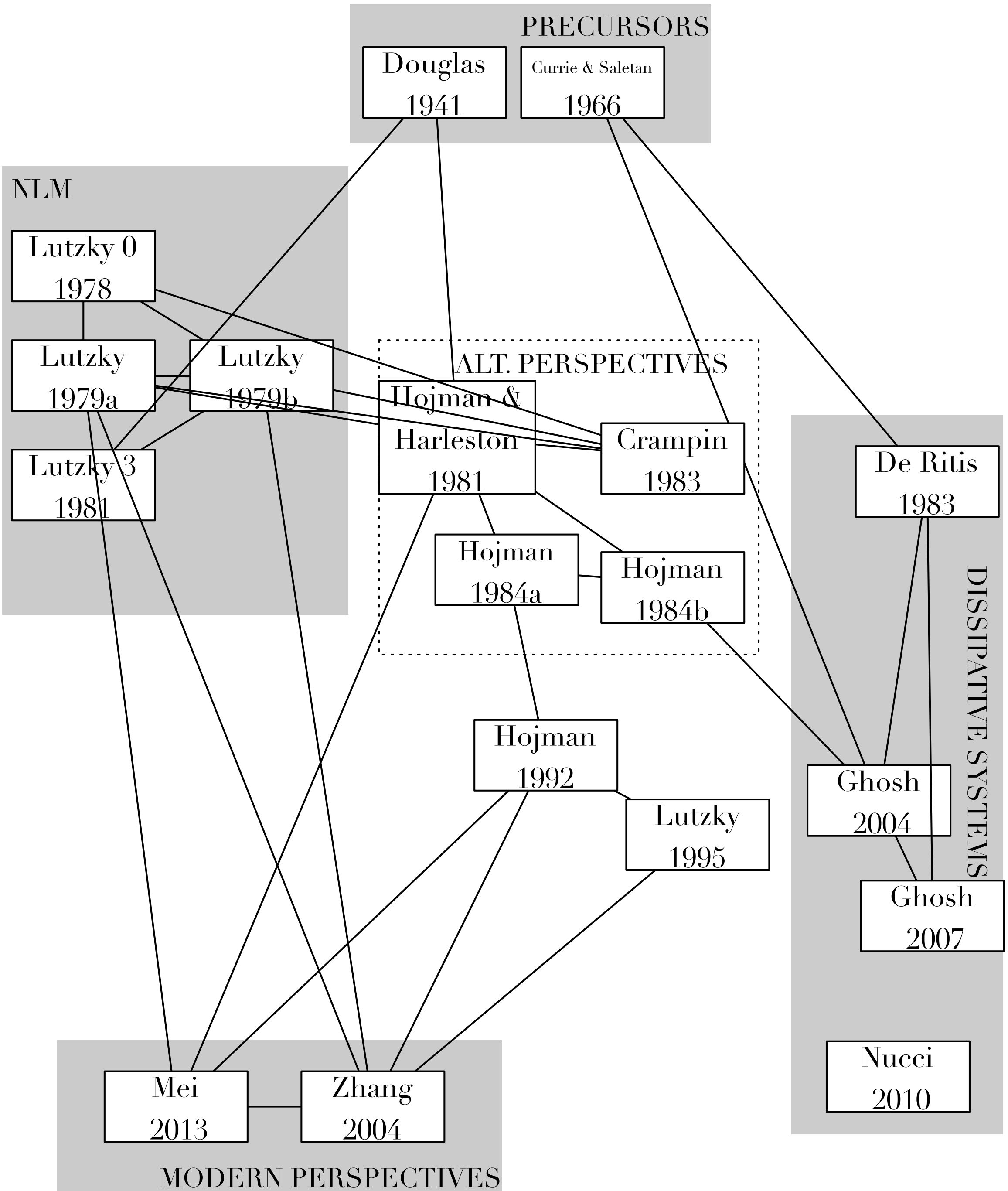

Figure A.1: Diagram of the relative relationship among the various authors that were concerned with the theory of Nonequivalent Lagrangian Mechanics. In this figure lines connecting article represent citations, which are ordered by noting that time flows roughly downwards.

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